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What is *the* Logic of Inference?

Abstract. The topic of this paper is the question whether there is a logic which could be justly called *the* logic of inference. It may seem that at least since Prawitz, Dummett and others demonstrated the proof-theoretical prominency of *intuitionistic* logic, the forthcoming answer is that it is this logic that is the obvious choice for the accolade. Though there is little doubt that this choice is correct (provided that *inference* is construed as inherently single-conclusion and complying with the Gentzenian structural rules), I do not think that the usual justification of it is satisfactory. Therefore, I will first try to clarify what exactly is meant by the question, and then sketch a conceptual framework in which it can be reasonably handled. I will introduce the concept of 'inferentially native' logical operators (those which explicate inferential properties) and I will show that the axiomatization of these operators leads to the axiomatic system of intuitionistic logic. Finally, I will discuss what modifications of this answer enter the picture when more general notions of inference are considered.

Keywords: inference, intuitionistic logic, proof theory, nature of logic, logical operators.

1. Introduction

Granting ourselves some oversimplification, we may say that logicians divide into model-theoreticians and proof-theoreticians. The former hold that it is model theory (or formal semantics) which is primary from the intuitive viewpoint (for it captures directly the *meanings* expressions have), whereas the latter hold that it is proof theory (or axiomatics) which is primary (for it captures the inferential patterns that are constitutive of what our words mean). In this paper I will not take sides (though there is no point in denying that my heart goes with the latter[1]), as I would prefer to concentrate on a question relevant not only to those who take special interest in proof theory, namely: "What is *the* logic of inference?"

The sense of the question is, to be sure, not quite clear; and it is to be expected that the very task of clarifying it will lead us some way towards its answer — an answer, moreover, which we may already know. For on what I take to be the most straightforward construal of the question, I would not

[1] All my cards are laid on the table elsewhere (see [20].)

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dispute with those theoreticians who claim that it is *intuitionistic* logic that is the logic most intimately related to inference. However, my aim here is to present what I believe is a deeper and a more general substantiation for it than is commonly on offer, and I will also discuss how this answer has to be adjusted in line with any shifts in how the question is interpreted.

2. Is there an "inferential logic"?

Let me begin by discussing an example of substantiation of the prominent role of intuitionistic logic *vis-à-vis* inferences and indicating why I think it fails. Let us consider a rather old, but still influential paper of Zucker and Tragesser [37], in which the authors claim to establish that it is the intuitionistic connectives which are "adequate for inferential logic". What exactly do they mean by this?

The authors stipulate that "given a logic L, and a set of logical operators of L, we say that S is *adequate for* L if every logical operation of L is explicitly definable in terms of S". In this sense we can obviously say that the set $\{\neg, \wedge\}$ of classical logical operators is adequate for the whole of the classical, truth-functional propositional calculus. But what about "inferential logic"? They say: "By 'inferential' logic we mean logic formulated in a natural deduction system, in which the meaning of each logical constant is supposed to be given by its set of *introduction rules*."

So the "adequacy problem" involves two sets of logical operators and a notion of equivalence: the task is to show that each of the first set can be equivalently rendered by those of the second set. In the case of classical logic the situation is simple: the first set is constituted by all the truth-functionally defined operators, the second one by $\neg$ and $\wedge$, and the equivalence in question consists in the fact that for any operator c of the first group there is a tautology of the classical propositional calculus of the form

$$c(S_1, \ldots, S_n) \leftrightarrow F,$$

where F is a formula containing no other operators than $\neg$ and $\wedge$.

Now, how is it with inferential logic? The first group of operators is obviously constituted by those which can be defined using the natural deduction rules, and Zucker and Tragesser explicitly narrow them down to those which can be defined using merely the introduction rules. But what is the second group, and, especially, what notion of equivalence should we consider?

The authors say that it is "the set of logical constants $\{\rightarrow, \wedge, \vee, \neg\}$" without specifying which version of the operators they mean (classical? intuitionist? any other?). This is rather surprising, especially in view of the fact that the claim is made to summarize the results of the paper; however, as the second class is to be a subclass of the first, we must assume that what the authors have in mind mean are $\to$, $\wedge$, $\vee$ and $\neg$ as defined within a natural deduction system and with some charity we can assume that what is meant is what is usually taken as the 'most natural' version of the definitions, i.e. the intuitionistic version. (The authors also appear to be willing to use the terms 'inferential logic' and 'intuitionistic logic' interchangeably).

But what is less clear is the notion of equivalence we should consider now. What one would expect is that for any operator c, definable in terms of the restricted natural deduction, there is a theorem of inferential logic of the form

$$c(S_1,\ldots,S_n) \leftrightarrow F,$$

where F is a formula containing only the intuitionistic $\to$, $\wedge$, $\vee$ and $\neg$. Again, it is unclear what the nature of the $\leftrightarrow$ is, but we will again assume that it is the intuitionistic operator.

Note that this solves the "adequacy problem of inferential logic" *within the medium of the very same logic*; hence it has compromised the possibility of interpreting the result as proving, in the intuitive sense, that it is intuitionistic logic that is *the* logic of inference. The obvious objection is that by using inferential logic as our metatheory we have smuggled in what was to have been discovered.

Moreover, the authors' argument raises further doubts. First, it is not entirely clear whether there is a deeper reason for their restricting themselves to operators defined by the standard form introduction rules, other than ease of treatment. They do not explain how they can adopt this restriction and at the same time continue talking about inferential logic in general.

Second, the actual way they carry out their proof also lacks clarity. Let us look at the first case they consider within the proof, the case where a propositional operator is defined by a single inferential rule:

$$\frac{\begin{array}{ccc} [A_{11},\ \ldots,\ A_{1k_1}] & & [A_{n1},\ \ldots,\ A_{nk_n}] \\ \vdots & \ldots\ldots\ldots\ldots\ldots\ldots & \vdots \\ B_1 & & B_n \end{array}}{\mathrm{c}(A_{11},\ \ldots,\ A_{1k_1},\ \ldots,\ A_{n1},\ \ldots,\ A_{nk_n},\ B_1,\ \ldots,\ B_n,\ C_1,\ \ldots,\ C_m)}$$

In this case, they simply state that "the meaning of c — or more exactly of $\mathrm{c}(\ldots,Aij,\ldots,B_i,\ldots,C_l,\ldots)$ — is given by

$(*)$ $(A_{11} \to B_1)\wedge\ldots\wedge(A_{1k_1} \to B_1)\wedge\ldots\wedge(A_{n1} \to B_n)\wedge\ldots\wedge(A_{nk_n} \to B_n)$".

The only substantiation they give of this claim is a reference to an assumption stated earlier in the paper, namely that "The meaning of the constant c is given by its set of introduction rules". Now, if the interpretation we have proposed for the authors' claims earlier are to be accepted, then it is, *prima facie*, strange to talk about "the meaning" of c. What we need to prove is that, given c is introduced by the rule above, there is a formula F containing as the only operators the intuitionistic $\rightarrow$, $\wedge$, $\vee$ and $\neg$, and such that

$$c(A_{11},\ldots,A_{1k_1},\ldots,A_{n1},\ldots,A_{nk_n},B_1,\ldots,B_n,C_1,\ldots,C_m) \leftrightarrow F.$$

And as F is apparently meant to be $(*)$, what we have to do is to prove, within the natural deduction system for intuitionistic logic, the formula

$$c(A_{11},\ \ldots,\ A_{1k_1},\ \ldots,\ A_{n1},\ \ldots,\ A_{nk_n},\ B_1,\ \ldots,\ B_n,\ C_1,\ \ldots,\ C_m) \leftrightarrow \\ ((A_{11}\wedge\ \ldots\ \wedge A_{1k_1}) \rightarrow B_1)\wedge\ \ldots\ \wedge((A_{n1}\wedge\ \ldots\ \wedge A_{nk_n}) \rightarrow B_n).$$

But surprisingly, they do not do anything like this. Can the reason be that the proof is too obvious? Well, the indirect implication is indeed pretty straightforward, but not so the direct one. It would amount to the transformation of the proof of $c(A_{11},\ldots,A_{1k_1},\ldots,A_{n1},\ldots,A_{nk_n},B_1,\ldots,B_n,C_1,\ldots,C_m)$ into the proof of $((A_{11}\wedge\ldots\wedge A_{1k_1})\rightarrow B_1)\wedge\ldots\wedge((A_{n1}\wedge\ldots\wedge A_{nk_n})\rightarrow B_n)$. It is clear how this is supposed to proceed: as the former is provable, and *as we assume that the only available proof is based on the introduction rule above* (which is what the authors express — oddly, in my view — by claiming that "the meaning of c is given by its introduction rule"), we must suppose that the premises of the proof are in force, and these premises establish the latter claim. This is a kind of 'abductive reasoning' which, though intuitively clear, should, I think, be given some explicit form.

Thus it would seem that Zucker and Tragesser's paper does not adequately establish an answer for the question posed as the title of this paper. And one important moral to draw from this discussion is that in order to do better we need a more explicit framework — a neutral ground where we can independently delimit both inferential systems and intuitionistic logic, and within which we can then show how the latter 'does justice' to the former.

3. The requirement of harmony

Of course that filling some of the gaps I have just diagnosed within Zucker's and Tragesser's paper is not too arduous. What appears to be implicit to their treatment of logic — though it does not quite surface — is the requirement of *harmony* between the ways logical operators are introduced and that in which they are eliminated. This requirement, endorsed already by Gentzen [10], was brought to the fore of logicians' attention especially by Dummett [7] and it is now one of the most discussed foundational issues of proof theory foundations. Let me present an argument provided by Neil Tennant [34]. Tennant first lists the desiderata for inferentially defined logical operators, and then shows that the desiderata are fulfilled by, and only by, intuitionistic operators. (In fact, he votes for a *relevant* version of intuitionistic logic, but we ignore this here.) The desiderata, according to him, are (p. 319):

(1) analytic systematicity,
(2) separability,
(3) immediacy,
(4) harmony of introduction and elimination rules,
(5) compositionality.

(1) amounts to the fact that each operator is introduced by means of a *finite number* of *schematic* rules; (2) to the fact that each of the rules contains merely a single dominant occurrence of an operator; (3) establishes that each rule is either an introduction rule, which contains the operator dominant in the conclusion, or an elimination rule, which contains it dominant in a premise; (4) that the introduction and elimination rules are in a precise sense complementary; and (5) that the rules have the subformula property.

Now it is clear that it is the harmony requirement which is tantamount to Zucker & Tragesser's requirement that "the meaning of a constant is given by its introduction rules". Tennant spells this out in much clearer terms:

> The introduction and elimination rules for any logical operator λ should be framed in such a way that (i) in the statement of the introduction rule for λ, the conclusion (with λ dominant) should be the strongest that can be inferred under the conditions specified; and (ii) in the statement of the corresponding elimination rule, the major premise (with λ dominant) should be the weakest that can be used in the way specified. (332–3)

The principle, as Tennant suggests, "serves to tailor the elimination rule to a previously chosen introduction rule, or vice versa". Hence if we encounter λ within a proof, we know it must have been introduced in its canonical way, and can reason back to the premises of its introduction rules, as done, in effect, by Zucker and Tragesser.

Hence Tennant shows that if we want logical operators which are not only inferentially introduced, but also in a certain sense well-behaved, we should restrict ourselves to the very ones Zucker and Tragesser are suggesting, namely those which are introduced by means of the standard form introduction rules (which are taken to 'implicitly contain' also the corresponding elimination rules). And he indicates that any such operators are reducible to the intuitionistic ones (which is what Zucker and Tragesser want to prove explicitly).

There is nothing to be objected to in this line of argumentation. It is based on the assumption that we should want a well-behaved logic — an assumption which is surely reasonable. But in this paper I would like to indicate that the question of *the* logic of inference might be answerable even without it. But before I turn to this, let me make a digression. I will sketch one possible answer to the question *what is logic?* — an answer that I deeply believe is correct[2], that I nevertheless will not argue for here. The reason why I am going to sketch it here is that it will help us clarify the question *what is the logic of inference?*.

4. Logical constants as tools of making inference explicit

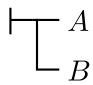
The answer to the question *what is logic?* I want to sketch is essentially due to Bob Brandom [3]: according to him, logical vocabulary of natural language, which logic strives to explicate, is first and foremost a tool for making explicit the rules implicit to our treating of concepts. According to Brandom, what underlies both human language and logic are inferences. Human language is structured in such a way that commitments to some claims bring about commitments to other claims — i.e. that the former ones entail the latter ones and thus the latter are correctly inferable from the former. (In fact, Brandom claims that there is a still deeper layer constituted by the concept of incompatibility.)

What we see as logical vocabulary is primarily a means of making explicit the proprieties implicit in our using language. Before we have "if ... then", the inference from "This is a dog" to "This is a mammal" can only be implicitly endorsed (or, as the case may be, violated), but once this connective is at hand, this inference can be explicitly expressed in the form of a claim, *viz.* "If this is a dog, then this is a mammal" (and with a more advanced logical vocabulary perhaps further transformed into "Every dog is a mammal"), and hence discussed w.r.t. its 'appropriateness' (and perhaps

[2] I have put forward some arguments in its favor elsewhere — see, e.g., [18].

in the end rejected). Thus logical vocabulary helps us make the rules that are intractably implicit in our practices explicit and discussable.

In this way, logical vocabulary appears as a means of 'internalizing the *meta*'. It allows us to say *within* a language what can otherwise be said only *about* the language — i.e. within a *meta*language. This Brandomian view of logic might look if not far-fetched, then at least at odds with what logic is usually supposed to be. Is it a mere philosophical jugglery which stems from a lack of sense for what (modern) logic really is? I would like to indicate (without giving a systematic account) that it is not; in particular that it is surprisingly consistent with the way logic was treated by some of its founding fathers[3]. Thus, consider Frege's ([8], p.5) famous introduction of material implication:

> Wenn A und B beurtheilbare Inhalte bedeuten, so gibt es folgende vier Möglichkeiten:
> 1) A wird bejaht und B wird bejaht;
> 2) A wird bejaht und B wird verneint;
> 3) A wird verneint und B wird bejaht;
> 4) A wird verneint und B wird verneint.
>
> 
>
>
> bedeutet nun das Urtheil, *dass die dritte dieser Möglichkeiten nicht stattfinde, sondern eine der drei andern.*[4]

Hence the implication is introduced to spell out, within his *Begriffsschrift*, what may otherwise only be articulable within a metalanguage. Before we have it, we can only abstain from denying B while asserting A (and possibly somehow sanction those who do not do so), but there is no way of bringing this practice into the open and making it subject to criticism. In contrast to this, once implication finds its way into the language, we are enabled to express the abstention by a statement (which can be challenged, justified etc.).

[3]I do not want to say that Brandom's proposal is unprecedented — to a certain extent it overlaps with the proof-theoretic approaches to logic (see, e.g. [30]), and especially with the German constructivist tradition [15]. But I think Brandom has given it the most thorough philosophical backing.

[4]"If A and B stand for contents that can become judgments, there are the following four possibilities: (1) A is affirmed and B is affirmed; (2) A is affirmed and B is denied; (3) A is denied and B is affirmed; (4) A is denied and B is denied. Now ... stands for the judgment that the third of these possibilities does not take place, but one of the three others does."

The situation is similar with respect to quantifiers. Frege (*ibid.*, p.19) writes

> In dem Ausdrucke eines Urtheils kann man die rechts von $\vdash$ stehende Verbindung von Zeichen immer als Funktion eines der darin vorkommenden Zeichen ansehen. *Setzt man an die Stelle dieses Argumentes einen deutschen Buchstaben, und giebt man dem Inhaltsstriche eine Hhlung, in der dieser selbe Buchstabe steht, wie in*
>
> $$\vdash\!\overset{\mathfrak{a}}{\smile}\!\!-\;\Phi(\mathfrak{a})$$
>
> *so bedeutet dies das Urtheil, da jene Function eine Thatsache sei, was man auch als ihr argument ansehen möge.*[5]

In this way, a quantified sentence becomes a shortcut for a *metalinguistic* statement: a statement about the results of replacing a part of an object language sentence by various (suitable) expressions. (Hence also to say that some natural language expressions, such as "something" or "everything", can be regimented by such quantifiers, is to say that these expressions are means of making some metalinguistic pronouncements, which hold *about* the language, explicit *within* the language.)

Russell ([28], p. 480) takes, in this respect, a very close train of thought:

> ... *everything* and *nothing* and *something* (...) are to be interpreted as follows:
> C(everything) means '$C(x)$ is always true';
> C(nothing) means '"$C(x)$ is false" is always true';
> C(something) means 'It is false that "C(x) is false" is always true'.

Hence again, sentences with "everything", "nothing" etc. are taken to express *metalinguistic pronouncements*; and their presence in a language thus enables us to say *in the language* what holds *about the language* and what is otherwise only expressible *within a metalanguage.*

Of course, the way from inferences to quantifiers is less straightforward than that from inferences to implication — for implication may be thought of as simply an internalization of inference, whereas the role of quantifiers within the explicitating of inferences is much more complex and inseparable from the role of some other constants. We will not deal with it in the current paper; it is discussed by Brandom ([3], esp. Chapter 7).[6]

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[5] "In the expression for a judgment, the complex symbol to the right of may always be regarded as a function of one of the symbols that occur in it. Let us replace this argument with a Gothic letter, and insert a concavity in the content-stroke, and make this same Gothic letter stand over the concavity, e.g.: .... This signifies the judgment that the function is a fact whatever we take its argument to be."

[6] Cf. also [19].

As I have already stressed, it it not my purpose here to try to vindicate this view of logic. But I think that it can show us a way to a clarification of the notion of "logic of inference" that is not only plausible, but also much more susceptible to an explication than the one that seems to animate the vague considerations of authors like Zucker and Tragesser. The clarification that I have in mind is the following: *The logic of inference is the logic of operators the principal purpose of which is making inference explicit.* In the next section, we start to build a formal framework in which this clarification will be given an unambiguous formal sense.

5. 'Inferentially native' operators

We have seen that the important precondition of dealing with the question of the logic of inference adequately is the establishment of a general enough framework. This is what we are about to do now. A *propositional language* will be a set S (of entities called *statements*) plus a finite set of operations on S. An *inferential structure* will be an ordered pair $I=\langle L, \vdash\rangle$, where $L=\langle S, \langle o_1, \ldots, o_n\rangle\rangle$ is a propositional language and $\vdash$ is a relation between finite sequences of elements of S and elements of S. Elements of S will be called the *statements of I* and $\vdash$ will be called the *inference relation of I.* (Note that as n may be 0, L may, in effect, be just a plain set.)

Given an inferential structure I, an ordered pair the first constituent of which (called the *antecedent*) is a finite sequence of statements and the second constituent of which (called the *consequent*) is a statement will be called an *inference* (of I). An inference will be called *valid* if its antecedent and its consequent stand in the relation $\vdash$. In this case the consequent will also be said to be *inferable* from the antecedent. A statement which is inferable from an empty sequence will be called a *theorem* (of I). A *metainference* over S will be an ordered pair the antecedent of which is a finite sequence of inferences and the consequent of which is an inference.

An *inferential rule* (or simply a *rule*), resp. a *metainferential rule* (a *metarule*), will be the name of an inference, resp. of a metainference with some parts replaced by parameters[7]. We will employ the letters A, A_1, A_2,

[7] We assume that (some of) the elements of S have ('proper') names, and the operations $o_1, \ldots, o_n$ allow us to form their further ('improper') names. Thus, if '$\otimes$' is a name of a binary operation and '$\mathbb{A}$' and '$\mathbb{B}$' are names of statements, then '$\mathbb{A} \otimes \mathbb{B}$' is a name of a statement, namely of the statement which results from the application of $\otimes$ to $\mathbb{A}$ and $\mathbb{B}$. (Note that hence '$\mathbb{A} \otimes \mathbb{B}$' is not the element itself, but a name; in particular the element itself need not consist of the three parts corresponding to the ones constituting the names.) Replacing some parts of an expression denoting an inference (such as $\langle\langle\mathbb{A}\otimes\mathbb{B}\rangle,\mathbb{B}\rangle$) by parameters we gain an inferential rule, and similarly for metainferences.

$\ldots, A_n, \ldots, B, C \ldots$ as parameters replacing names of statements, and $X, Y, Z, \ldots$ as those replacing finite sequences of names of statements. (Hence '$\langle\langle A\otimes B\rangle, B\rangle$' will be an inferential rule, and '$\langle\langle\langle\langle A\rangle, C\rangle, \langle\langle B\rangle, C\rangle\rangle, \langle\langle A\otimes B\rangle, C\rangle\rangle$' a metainferential rule. The quotes will be omitted.) An inferential rule with an empty antecedent will be called an *axiom.* An *instance of a rule* (*metarule*) *over S* is any inference (metainference) over S whose name can be obtained from the rule (metarule) by a systematic replacement of parameters by appropriate names. (Thus, the set of instances of a rule (metarule) constitutes a function and the rule (metarule) can also be identified with this function.)

What will interest us from our logical perspective are especially inferential structures generated by finite means (and especially ways of building such structures). To delimit these, we need more conceptual machinery. An *inferential basis* will be an ordered triple $\langle L, R, M\rangle$, where $L = \langle S, O\rangle$ is a propositional language, R is a finite set of inferential rules over S, and M is a finite set of metainferential rules over S. The notion of *inference valid* over a basis is defined in the following obvious recursive way:

(i) an instance of an inferential rule from R is valid;
(ii) if $\langle\langle R_1, \ldots, R_n\rangle, R\rangle$ is an instance of a metainferential rule from M and all of $R_1, \ldots, R_n$ are valid, then so is R;
(iii) nothing else is a valid inference.

We will say that a basis *generates* the structure consisting of L plus the relation constituted by all the inferences valid over it. Elements of R will be called the *basic rules* of the basis, whereas those of M will be called its *basic metarules.* Two inferential bases with the same underlying language will be called *equivalent* iff they generate identical structures.

An inferential rule will be called *admissible* for a structure I iff all its instances are valid in I. If $\langle X, A\rangle$ is admissible for I, then we will write $X \vdash_I A$; and we will leave out the subscript accompanying $\vdash$ where no confusion would be likely to arise. A metainference is *valid* in I iff either at least one of the inferences in its antecedent is not valid in I, or the inference in its consequent is. A metarule is *admissible for I* iff all its instances are valid in I. If $\langle\langle R_1, \ldots, R_n\rangle, R\rangle$ is a metarule admissible for I, then we will write

$$\frac{R_1, \ldots, R_n}{R}{}_I$$

and we will again leave out the subscript where feasible. It is clear that all basic rules and all basic metarules of a basis are admissible for the structure generated by the basis. Suppose now that B is inferable from A, i.e. that

$$A \vdash B.$$

What would it mean to make this fact *explicit* within the underlying structure? We need a statement which *says* that B is inferable from A. But what does it take for a statement of such a structure to *say* this? Presumably to be true iff B is inferable from A. But the relation $\vdash$ is unchanging and hence the explicitating claim would be true necessarily; and the counterpart of necessary truth within the structure is clearly theoremhood.

Hence to make the inferability of a statement from another statement explicit is to have, for every pair of statements A and B, a statement which is a theorem iff $A \vdash B$. Let us form the name of such an 'explicitating' statement by means of the sign $\triangleright$, hence let, for every A and B,

$$(*) \quad A \vdash B \text{ iff} \vdash A \triangleright B.$$

We will call the operator defined in this way a *deductor* (for the inferential structure). (Note the indefinite article; $(*)$ can be obviously satisfied by rather different operators.) Given this, to claim $A \triangleright B$ (as a necessary truth, i.e. $\vdash A \triangleright B$) *is* to claim that B is inferable from A.

It is clear that $(*)$ is valid for every A and B iff the following two metarules are admissible:

$$\text{(DED)} \quad \frac{A \vdash B}{\vdash A \triangleright B} \qquad\qquad \text{(CODED)} \quad \frac{\vdash A \triangleright B}{A \vdash B}$$

This yields us also the answer to the question of how to *build* a structure with a deductor: it is clearly enough to have the binary operator and to include (DED) + (CODED) into the basis.

However, $\triangleright$ allows us to say that a statement is inferable from another statement, but not yet that it is inferable from a *sequence* of statements. An obvious way how to make this explicit is by way of introducing, for every pair of statements A and B, another new statement, say $A \otimes B$, such that

$$X, A, B, Y \vdash C \text{ iff } X, A \otimes B, Y \vdash C,$$

Let us call the new operator $\otimes$ the *amalgamator*. The definition of amalgamator can again be given in terms of admissibility of a pair of metarules:

$$\text{(AMLG)} \quad \frac{X, A, B, Y \vdash C}{X, A \otimes B, Y \vdash C} \qquad\qquad \text{(DEAMLG)} \quad \frac{X, A \otimes B, Y \vdash C}{X, A, B, Y \vdash C}$$

If X is the sequence $A_1 A_2 \ldots A_{n-1} A_n$, then we will write $\otimes X$ as the shorthand for $(A_1 \otimes (A_2 \otimes (\ldots (A_{n-1} \otimes A_n))))$. Now it is obviously the case that

$$X \vdash A \text{ iff } \vdash (\otimes X) A$$

However, there is also an alternative to the introduction of the amalgamator; we can also make the deductor recursive, by requiring

$$(**)\ X, A \vdash B \text{ iff } X \vdash A \rhd B,$$

i.e. the admissibility of

$$\text{(DED*)} \quad \frac{X, A \vdash B}{X \vdash A \rhd B} \qquad \text{(CODED*)} \quad \frac{X \vdash A \rhd B}{X, A \vdash B}$$

Hence both the deductor and the amalgamator are operators which emerge as natural tools once we set out to make the relation of inference explicit[8]. Are there some other similarly ‘inferentially native’ operators?

The ones we have introduced so far let us explicitate claims to the effect that a statement is inferable from other statements. But we might also want to claim the contrary: namely that a statement is *not* inferable from other statements. If we write $X \nvdash A$ for “A is not inferable from X”, then we might want to have an ‘anti-deductor’ $\not\rhd$ such that

$$X, A \nvdash B \text{ iff } X \vdash A \not\rhd B.$$

However, in contrast to the previous cases, it is wholly unclear how this could be turned into inferential (meta)rules which could be integrated into a basis for an inferential system. Therefore, we leave the matter at this for now and we will return to it later.

6. Standard inferential structures

We have defined inferential structure simply as a propositional language plus any kind of relation between finite sequences of statements and statements; and we have restricted our attention to the finitely generated structures. (What our framework is supposed to explicate is the fact that language-using creatures would stipulate — not necessarily by an explicit convention, possibly by an establishment of a praxis — inferential rules and thereby establish inferential relationships. This means that what we should concentrate on are structures generated by finite collections of rules.) However, should we not be even more restrictive?

It seems that we are not interested in relations generated from the basic rules in just *any* way, but rather in the definite relation of ‘provability by means of the basic rules’. And the relation of *provability by means of the collection R of rules* is precisely the admissibility of the Gentzenian *structural metarules*[9]:

[8]They are what Avron [1] called the internal implication and conjunction, respectively.

(REF)	$\overline{A \vdash A}$
(EXT)	$\dfrac{X, Y \vdash A}{X, B, Y \vdash A}$
(CON)	$\dfrac{X, A, A, Y \vdash B}{X, A, Y \vdash B}$
(PERM)	$\dfrac{X, A, B, Y \vdash C}{X, B, A, Y \vdash C}$
(CUT)	$\dfrac{X, A\ Y \vdash B \qquad Z \vdash A}{X, Z, Y \vdash B}$

Again, let us make things more precise. Let S be a set and R a collection of inferential rules over S. A *proof* of A from X by means of R (where $A \in S$ and X is a finite sequence of elements of S) is defined in the expected way: it is a finite sequence of elements of S which ends with A and whose every element is either an element of X or is the value of a rule from R for some arguments which all occur within the sequence earlier. If there is a proof of A from X, then A will be said to be *provable from* X. Given a basis, we will say that A is provable from X over it iff A is provable from X by means of its basic rules.

Now we can formulate:

THEOREM 6.1.[10] *A is provable from X by means of R iff $X \vdash A$ is derivable from R by means of (REF), (EXT), (CON), (PERM) and (CUT).*

PROOF. Let A be provable from X by means of R. Then there is a sequence $A_1, \ldots, A_n$ of statements such that $A_n = A$ and every A_i is either an element of X or is inferable by a rule from R from statements which are among $A_1, \ldots, A_{i-1}$. If $n=1$, then there are two possibilities: either $A \in X$ and then $X \vdash A$ in force of (REF) and (EXT); or A is an axiom (i.e. a rule with an empty antecedent) of R, and then $\vdash A$ and hence $X \vdash A$ in force of (EXT). If $n>1$ and A_n is inferable from some $A_{i_1}, \ldots, A_{i_m}$ (where $i_1<n, \ldots, i_m<n$) by a rule from R, then $A_{i_1}, \ldots, A_{i_m} \vdash A$, where $X \vdash A_{i_j}$ for $j=1, \ldots, m$.

[9] The reason why we take also the first one as a metarule (with the empty antecedent), rather than a rule, is purely simplicity of exposition. Nothing relevant for our considerations is affected by this move. Note that a metarule with an empty antecedent is admissible just in case its consequent is an admissible axiom. Hence, an inferential basis ⟨L, R, M⟩ is equivalent to ⟨L, ∅, M∪R*⟩, where R* consists of metarules with empty antecedents and the elements of R as consequents.

[10] Versions of this theorem were proved by various authors; see, e.g. [29].

Then $X,\ldots,X \vdash A$ in force of (CUT), and hence $X \vdash A$ in force of (PERM) and (CON).

Now, conversely, let $X \vdash A$ be derivable from R by means of (REF), (EXT), (CON), (PERM) and (CUT). We will show that A is provable from X by induction. If $X \vdash A$ is an instance of a rule from R, then its provability from X is obvious. Now assume that $X \vdash A$ is derivable by one of (REF), (EXT), (CON), (PERM) and (CUT) from inferences for which the inductive assumption holds. Suppose that it is derivable by (EXT). Then it must be the case that $X \vdash A$ is derivable from a $X' \vdash A'$ such that A' is provable from X' by means of R, A' is A, and all elements of X' are elements of X. But then the proof of A' from X' is also the proof of A from X. The case of the other structural metarules is equally easy. ∎

We will say that an inferential structure is *standard* if it admits the structural rules (i.e. (REF), (EXT), (CON), (PERM) and (CUT)). We will say that the basis is *standard* iff its set of basic metarules contains the structural rules; we will say that it is *strictly standard* iff the set coincides with the set of the five structural rules. (It is clear that a structure generated by a standard base is standard.) Using this terminology, we can say that the bases we should concentrate on are the strictly standard ones. A structure generated by the strictly standard structure with the set R of basic rules will be called *based on R*.

Note that Theorem 6.1 indicates that a strictly standard basis thus amounts to what is usually seen as an *axiomatic system* or a *Hilbertian calculus*. It is based on a collection of axioms (those of its inferential rules which have an empty antecedent) plus a collection of rules (the rest of inferential rules); and its theorem are those statements which are probable, in the usual sense, from the axioms by means of the rules.

If we now want to build a strictly standard inferential basis establishing the native inferential operators (hence an axiomatization of the logic of inference), we need to characterize all the operators by rules (rather than metarules). Our next task, hence, is to try to turn the metarules we have used to characterize the operators into rules. Before we take it up, let us note a corollary of Theorem 6.1, namely that within certain inferential structures, provability comes to coincide with inferability:

Corollary 6.2. *In a structure generated by a strictly standard base with (MP) admissible, A is provable from X iff* $X \vdash A$.

Proof. Let A be provable from X by means of the basic rules of the structure. Then, in force of Theorem 1, the inference $\langle X,A\rangle$ is derivable from these rules by means of the structural metarules and hence is valid. Let, conversely, $X \vdash A$. Then A is provable from X by means of (MP). ■

7. Deductor

Let us start with the deductor. It is easily seen that if we restrict ourselves to standard structures, (CODED*) becomes equivalent to the *modus ponens* inferential rule:

$$\text{(MP)} \quad A \triangleright B, A \vdash B.$$

This is spelled out by the following theorem:

THEOREM 7.1. *Within a standard inferential structure, (CODED*) is admissible iff (MP) is.*

PROOF.

<u>(CODED*) ⇒ (MP)</u>

1. $A \triangleright B \vdash A \triangleright B$ (REF)
2. $A \triangleright B, A \vdash B$ from 1 by (CODED*)

<u>(MP) ⇒ (CODED*)</u>

1. $X \vdash A \triangleright B$ assumption
2. $A \triangleright B, A \vdash B$ (MP)
3. $X, A \vdash B$ from 1,2 by (CUT) ■

However, things are not that easy with (DED*): there does not appear to be an inferential rule or a collection of inferential rules which would be equivalent to it. However, for a structure generated by a strictly standard basis there *is* nevertheless a possibility of securing the admissibility of (DED*) by stipulating inferences. The point is that as what (DED*) claims is that whenever $X, A \vdash B$, then also $X \vdash A \triangleright B$, it would be enough to stipulate $X \vdash A \triangleright B$ for every particular case when $X, A \vdash B$. And as for a structure generated by a strictly standard basis, $X, A \vdash B$ just in case this is an instance of a basic rule, or it is derivable from basic rules by means of the structural metarules, it would be enough to stipulate this for every basic rule and to secure that this property is preserved by all metarules.

The *internalization* of the inference $A_1, \ldots, A_n \vdash B$ will be the statement $A_1 \triangleright (A_2 \triangleright (\ldots (A_n \triangleright B)))$, which we will also abbreviate as $A_1, \ldots, A_n \blacktriangleright B$. (The internalization of an inference with the empty antecedent will be the very same inference.) An inference $A_1, \ldots, A_n \vdash B$ will be called *internalized* if its internalization is a theorem, i.e. iff $\vdash A_1, \ldots, A_n \blacktriangleright B$. A rule is *internalized* if all its instances are. Given this, we can prove the following lemma:

Lemma 7.2. *(DED*) is admissible in a structure generated by an inferential basis with (MP) admissible iff each of its basic rules is internalized and each of its basic metarules preserves internalizedness, i.e. maps internalized inferences always also only on inferences that are internalized.*

Proof. Let (DED*) be admissible. Then all valid inferences are internalized. Thereby, also all admissible rules are internalized (because a rule is internalized iff all its instances are). Moreover, as $\vdash A_1,\ldots,A_n \blacktriangleright B$ yields, via (MP), the provability of B from $A_1,\ldots,A_n$, also all internalized inferences are, in force of Corollary 6.2, valid and hence all internalized rules are admissible. Hence also all inferences on which metarules map internalized inferences are internalized: every internalized inference is, we saw, valid, and hence it is bound to be mapped, by a metarule, on a valid, and hence an internalized inference.

Conversely, if all basic rules are internalized and all basic metarules preserve internalizedness, then *every* valid inference is internalized; and hence (DED*) holds. ■

What does it take for the structural metarules to preserve internalizedness? The following lemma offers an answer:

Lemma 7.3. *The structural metarules preserve internalizedness iff (A1)-(A5) are admissible:*

(A1) $\vdash A \triangleright A$

(A2) $\dfrac{\vdash B}{\vdash A \triangleright B}$

(A3) $\dfrac{\vdash A, A \blacktriangleright B}{\vdash A \triangleright B}$

(A4) $\dfrac{\vdash X, A, B, Y \blacktriangleright C}{\vdash X, B, A, Y \blacktriangleright C}$

(A5) $\dfrac{\vdash X \blacktriangleright A \vdash A \triangleright B}{\vdash X \blacktriangleright B}$

Proof. A straightforward transcription of what it takes for (REF), (EXT), (CON), (PERM) and (CUT) to preserve internalizedness yields:

(A1) $\vdash A \triangleright A$

(A2*) $\dfrac{\vdash X, Y \blacktriangleright B}{\vdash X, A, X \blacktriangleright B}$

(A3*) $\dfrac{\vdash X,A,A,Y \blacktriangleright B}{\vdash X,A,Y \blacktriangleright B}$

(A4) $\dfrac{\vdash X,A,B,Y \blacktriangleright C}{\vdash X,B,A,Y \blacktriangleright C}$

(A5*) $\dfrac{\vdash X \blacktriangleright A \qquad \vdash Y,A,Z \blacktriangleright B}{\vdash Y,X,Z \blacktriangleright B}$

Now it is easy to see that in view of (A4), we can reduce (A2*), (A3*) and (A5*) to the simpler (A2), (A3) and (A5), respectively. ■

However, from the viewpoint of our aim, namely reduction of metarules to rules, Lemma 7.3 is not yet very useful. (A2)-(A5) are still *meta*rules; so we are as yet no better off than with DED* itself. The following theorem gives rules which are capable of guaranteeing the admissibility of these metarules:

LEMMA 7.4. *If a standard inferential structure admits (MP) plus*
(▷1) $\vdash B \triangleright (A \triangleright B)$
(▷2) $\vdash (A \triangleright (A \triangleright B)) \triangleright (A \triangleright B)$
(▷3) $\vdash (C \triangleright A) \triangleright ((A \triangleright B) \triangleright (C \triangleright B))$,
then (A1)-(A5) are admissible in it.

PROOF. It is clear that the admissibility of (A2), (A3), (A4) and (A5) is secured by
(B2) $B \vdash A \triangleright B$;
(B3) $A, A \blacktriangleright B \vdash A \triangleright B$;
(B4) $X,A,B,Y \blacktriangleright C \vdash X,B,A,Y \blacktriangleright C$; and
(B5) $X \blacktriangleright A,\ A \triangleright B \vdash X \blacktriangleright B$,
and hence by
(C2) $\vdash B \triangleright (A \triangleright B)$;
(C3) $\vdash (A,A \blacktriangleright B)\ (A \triangleright B)$;
(C4) $\vdash (X,A,B,Y \blacktriangleright C) \triangleright (X,B,A,Y \blacktriangleright C)$; and
(C5) $\vdash (X \blacktriangleright A) \triangleright ((A \triangleright B) \triangleright (X \blacktriangleright B))$.
Now it can easily be seen that (C4) can be reduced to (C4*) and (C5) to (C5*):
(C4*) $\vdash (A,B \blacktriangleright C)(B,A \blacktriangleright C)$;
(C5*) $\vdash (C \triangleright A) \triangleright ((A \triangleright B) \triangleright (C \triangleright B))$;
and that (A1) and (C4*) then follow from (C2), (C3) and (C5*). ■

Hence, given a structure generated by a strictly standard inferential basis, if (▷1), (▷2) and (▷3) are admissible for it, then (DED*) is also admissible once all its basic inferences are internalized. This means that we can make

$\triangleright$ into a deductor by stipulating (MP), ($\triangleright$1), ($\triangleright$2) and ($\triangleright$3), and by replacing every basic rule $X \vdash A$ by $\vdash X \blacktriangleright A$. (As $\vdash X \blacktriangleright A$ yields $X \vdash A$ via (MP), once we add the former, we can delete the latter.)

Thus we have seen that within a structure generated by a strictly standard inferential basis, $\triangleright$ is a deductor if (MP), ($\triangleright$1), ($\triangleright$2), ($\triangleright$3) and the internalizations of all basic rules are admissible. Now we will show that it is a deductor *only if* this condition is fulfilled.

LEMMA 7.5. *If $\triangleright$ is a deductor of an standard inferential structure, then ($\triangleright$1), ($\triangleright$2) and ($\triangleright$3) are admissible for the structure.*

PROOF. Let us assume that one of ($\triangleright$1), ($\triangleright$2) and ($\triangleright$3) is not admissible. Let it be, for example, ($\triangleright$2) (the other cases are analogous). As $\triangleright$ is a deductor, (MP) is admissible and hence B is provable from $A\triangleright(A\triangleright B)$ and A. Hence, in view of Corollary 6.2,

$$A\triangleright(A\triangleright B), A \vdash B.$$

(2) now follows by means of (DED*). ■

THEOREM 7.6. *$\triangleright$ is a deductor of an inferential structure generated by a strictly standard base iff the structure admits (MP), ($\triangleright$1), ($\triangleright$2) and ($\triangleright$3) and all its basic rules are internalized.*

PROOF. Let $\triangleright$ be a deductor. Then all its basic rules are internalized in force of (DED*). Moreover, in force of Theorem 7.1, the structure admits (MP). Also, in force of Lemma 7.5, it admits ($\triangleright$1), ($\triangleright$2) and ($\triangleright$3).

Conversely, let the structure admit (MP), (1), (2) and (3) and let all its basic rules be internalized. Then, in force of Theorem 7.1, it admits (CODED*). Also, in force of Lemma 7.4, it admits (A1)-(A5), and hence, in force of Lemma 7.3, its structural metarules preserve internalizedness. Hence, as the structural metarules are the only metarules and as all the rules are internalized, (DED*) is admissible. ■

If we now look at $\triangleright$ as at an implication, then we can see that what we have reached in this way is the axiomatization of the so called *positive logic*[11]. It constitutes an axiomatization of the purely implicative part of the intuitionistic propositional calculus; and also of the purely implicative part of the classical propositional calculus — if "implicative part" is interpreted as referring to what is provable from purely implicative axioms. (Should it be interpreted as referring to all purely implicative theorems, then the situation would be different, for within classical logic, unlike within intuitionistic

[11]Hilbert and Bernays ([13], Supplement III).

logic, negation is not conservative over implication, and hence the class of implicative theorems exceeds the class of statements provable from purely implicative axioms.)

8. Amalgamator

Now let us consider the amalgamator. It is not difficult to show that within a standard inferential structure, (AMLG) becomes equivalent to (ICN), whereas (DEAMLG) to (ECN1) plus (ECN2):

(ICN)	$A,B \vdash A\otimes B$
(ECN1)	$A\otimes B \vdash A$
(ECN2)	$A\otimes B \vdash B$

THEOREM 8.1. *For a standard inferential structure, (AMLG) and (DEAMLG) are admissible iff (ICN), (ECN1) and (ECN2) are.*

PROOF.

(ICN) $\Rightarrow$ (DEAMLG)
1. $X,A\otimes B,Y \vdash C$ assumption
2. $A,B \vdash A\otimes B$ (ICN)
3. $X,A,B,Y \vdash C$ from 2 and 1 by (CUT)

(DEAMLG) $\Rightarrow$ (ICN)
1. $A\otimes B \vdash A\otimes B$ (REF)
2. $A,B \vdash A\otimes B$ from 1 by (DEAMLG)

(ECN) $\Rightarrow$ (AMLG)
1. $X,A,B,Y \vdash C$ assumption
2. $A\otimes B \vdash A$ (ECN1)
3. $A\otimes B \vdash B$ (ECN2)
4. $X,A\otimes B,A\otimes B,Y \vdash C$ from 1, 2, 3 by (CUT)
5. $X,A\otimes B,Y \vdash C$ from 4 by (CON)

(AMLG) $\Rightarrow$ (ECN1)
1. $A \vdash A$ (REF)
2. $A,B \vdash A$ from 1 by (EXT)
3. $A\otimes B \vdash A$ from 2 by (AMLG)

(AMLG) $\Rightarrow$ (ECN2)
1. $A \vdash A$ (REF)
2. $A,B \vdash B$ from 1 by (EXT)
3. $A\otimes B \vdash B$ from 2 by (AMLG) ■

Now if we have also the deductor, we have a way of making all inferential rules, with the exception of *modus ponens*, explicit, i.e. of reducing them to *axioms*. (And, in fact, we *must* do so in order to keep (DED*) valid.) As for those constitutive of the amalgamator, i.e. (ICN), (ECN1) and (ECN2), they are transformed into

$$
\begin{array}{ll}
(\otimes 1) & \vdash A \rhd (B \rhd (A \otimes B)) \\
(\otimes 2) & \vdash A \otimes B \rhd A \\
(\otimes 3) & \vdash A \otimes B \rhd B
\end{array}
$$

9. Anti-deductor?

We have noticed that besides the deductor and the amalgamator, which play their respective parts in explicitating inferability, we might consider an operator which would explicitate *non*inferability just like the deductor explicitates inferability,

$$X, A \nvdash B \text{ iff } X \vdash A \not\rhd B,$$

but we have not seen any way of transforming this into the kind of rules or metarules we are after.

Moreover, such an operator would not be feasible at all. It is clear that non-deducibility does not admit weakening, in the sense that a conclusion's not being deducible from premises surely does not entail its not being deducible from *more* premises. But the presence of the antideductor would force just this: if $X, A \nvdash B$ yields $X \vdash A \not\rhd B$, then it yields also $X, C \vdash A \not\rhd B$, and hence $X, A, C \nvdash B$. (In particular, if $A \nvdash B$, then $A, B \nvdash B$, which is hardly what we could accept[12].)

It follows that the fact that a statement is not inferable from other statements should not be a premise of the introduction rule of a logical operator. But what might still be possible is to consider a weakened version of the project of an anti-deductor, which would not feature non-inferability in this problematic way. We can consider the possibility that what we will make explicit in terms of $A \not\rhd B$ would be not that B is not inferable from A, but that B *cannot* become inferable from A.

But could this happen at all? Are we in some cases warranted in requiring that an inferential link between A and B cannot be forged as a matter of principle? Well, a situation which we should surely want to avoid is a breakdown of the whole inferential structure. Hence if an extension of an inferential relation could bring about such a breakdown, we had better block it. Can this happen? Can an inferential structure 'break down'?

[12] I owe this observation to Greg Restall.

An inferential structure is usually constructed with the goal of capturing inferential links making up a real language. The goal, however, may not be achieved: the structure may fail to capture the intended language adequately. But of course this does not prevent it from capturing another (real or possible) language. Is there a way in which an inferential structure can be a *complete* failure?

It would surely be such a failure if it were *trivial.* A structure with an empty inference relation is, from this viewpoint, clearly anomalous. And the same holds for a structure in which everything is inferable from everything. Structures like these are clearly worthless; and we should avoid turning ours into one such. Hence the situation in which making A inferable from X would result in making everything inferable from everything (it is clear that it cannot result in making nothing inferable from nothing!) should make us block the inference. Therefore, writing "$X \vdash \bot$" for *everything is inferable from* X (or X *is inconsistent*), we might, coming back to our vague notion of an anti-deductor, want at least

$$\text{(A5*)} \quad \frac{B \vdash \bot \qquad \vdash A}{\vdash A \ntriangleright B}$$

and more generally

$$\frac{X,B \vdash \bot \qquad X \vdash A}{X \vdash A \ntriangleright B}.$$

(As for the latter, it is important to realize that the intuitive sense of $X \vdash A \ntriangleright B$ is *not* "B should not be inferable from X and A", but rather "provided all elements of X are theorems, B should not be inferable from A".)

Moreover, it seems that if $B \vdash \bot$, we should also have to block the very possibility of $\vdash B$, wherefore we would need not an anti-deductor, but a unary operator $\varnothing$ such that

$$\frac{B \vdash \bot}{\vdash \varnothing B}$$

and, more generally

$$\frac{X,B \vdash \bot}{X \vdash \varnothing B}$$

(Again, it is important to realize that $X \vdash \varnothing B$ should not be read as "B should not be inferable from X", but rather "if all elements of X are theorems, then B should not be a theorem".)

Let us call an operator marking potential inferences which would lead to the fatal explosion of the inference relation an *explosion-detector*. Hence an explosion-detector $\varnothing$ is governed not only by

$$(\text{ED}) \quad \frac{X, B \vdash \bot}{X \vdash \varnothing B}$$

but also by the converse

$$(\text{COED}) \quad \frac{X \vdash \varnothing B}{X, B \vdash \bot}$$

It is easy to see that once we have an explosion-detector $\varnothing$ and a deductor, we can define a (sort of) anti-deductor in their terms: $A \ntriangleright B$ can be a shorthand for $\varnothing(A \triangleright B)$.

However, let us stress we are still only half way to the definition of the explosion-detector: '$X \vdash \bot$' does not denote an inferential rule, it is merely our shortcut for '$X \vdash A$ for every A'. In some contexts it would be possible to replace '$X \vdash \bot$' simply by '$X \vdash A$' (where A is not a constituent of X), but this would clearly not work in the position of the antecedent of a metarule:

$$\frac{X, B \vdash A}{X \vdash \varnothing B}$$

does *not* state that if everything is inferable from X and B, then $\varnothing B$ is inferable from X, but rather that this is the case if *anything* (i.e. at least one statement) is inferable from X and B. Also it is not possible to replace '$X \vdash \bot$' by all inferences of the form $X \vdash A$ — for these are infinite in number (unless our language is finite, which is clearly not an interesting case).

There is the well-known easy way out of this: namely to start to see '$\bot$' as a new logical constant ('nullary operator') characterized by the rule

$$(\text{EXPL}) \quad \bot \vdash A$$

Given this, we can construe (ED) and (COED) as fully fledged metarules[13].

[13]This is, in a sense, a sleight of hand – a more straightforward way would be to accept incompatibility as a new primitive concept and '$X \vdash \bot$' as a new piece of primitive notation (then perhaps better written as '$\bot X$' — cf. [21]). As Tennant [33] argues, since inference is to record truth-transmission, $X \vdash A$ makes nontrivial sense only if there is something to transmit from X to A, i.e. if X is capable of being true at all. From this vantage point, marking inconsistency is a task naturally instrumental to the task of marking truth-transmission, i.e. inference; and hence the introduction of an inconsistency marker is a natural continuation of introducing a deductor. Moreover Brandom ([4], Chapter 6), following Wilfrid Sellars, claims that the concept of inconsistency, or the interdefinable concept of incompatibility, should be seen as more fundamental than the concept of inference. This is because the 'language game' of giving and asking for reasons that gives rise to logical vocabulary is fuelled by the fact that we can both give reasons ourselves, i.e. display statements from which a given statement is inferable, and also challenge others to do so, i.e. display statements which are incompatible with the given one. (To see the concept of incompatibility as primary w.r.t. that of inference is then to say that A is correctly inferable from X iff whatever is incompatible with X is also incompatible with A.)

(It is clear that once we do this, (ED) and (COED) become the respective instances of (DED) and (CODED) so that the whole ‘work of explosion-detecting’ becomes loaded on $\bot$.)

Within the framework of a standard inferential structure, we can reduce (COED) to

$$\text{(COED*)}\quad \varnothing A, A \vdash B:$$

THEOREM 9.1. *Within a standard inferential structure, (COED) is admissible iff (COED*) is.*

PROOF.

(COED*) $\Rightarrow$ (COED)

1. $X \vdash \varnothing A$	assumption
2. $X, A \vdash B$	from 1 and (COED*) by (CUT)
3. $X, A \vdash \bot$	from 2.

(COED) $\Rightarrow$ (COED*)

1. $\varnothing A \vdash \varnothing A$	(REF)
2. $\varnothing A, A \vdash \bot$	from 1 by (COED)
3. $\varnothing A, A \vdash B$	from 2. and (EXPL) by (CUT) ■

Moreover, in the presence of a deductor, we can reduce (ED) to (ED*) (to simplify formulas, we accept the usual assumption that unary operators take precedence over binary ones):

$$\text{(ED*)}\quad A \rhd B, A \rhd \varnothing B \vdash \varnothing A$$

THEOREM 9.2. *Within a standard inferential structure which admits (COED*), (ED) is admissible iff (ED*) is.*

PROOF.

(ED*) $\Rightarrow$ (ED)

1. $X, A \vdash B$	from the assumption
2. $X \vdash A \rhd B$	from 1 by (DED*)
3. $X, A \vdash \varnothing B$	from the assumption
4. $X \vdash A \rhd \varnothing B$	from 3 by (DED*)
5. $X, X \vdash \varnothing A$	from 2, 4 and (ED*) by (CUT)
6. $X \vdash \varnothing A$	from 5 by (PERM) and (CON)

(ED) $\Rightarrow$ (ED*)

1. $A \rhd B, A \vdash B$	(MP)
2. $A \rhd \varnothing B, A \vdash \varnothing B$	(MP)
3. $A \rhd \varnothing B, A, AAB, A \vdash$	from 1, 2 and (COED*) by (CUT)
4. $AAB, A \rhd \varnothing B, A \vdash$	from 3 by (PERM) and (CON)
5. $A \rhd B, A \rhd \varnothing B \vdash \varnothing A$	from 4 by (ED) ■

Finally, we must internalize (COED*) and (ED*), i.e. recast them as

$$\begin{array}{ll} (\varnothing 1) & \vdash \varnothing \mathrm{A} \triangleright (A \triangleright B) \\ (\varnothing 2) & \vdash (A \triangleright B) \triangleright ((A \triangleright \varnothing \mathrm{B}) \triangleright \varnothing \mathrm{A}) \end{array}$$

If we now summarize the inferential rules which are required to constitute our three native inferential operators within the context of a standard inferential structure, we can say that $\triangleright$ is a deductor, $\otimes$ is an amalgamator and $\varnothing$ is an explosion-detector iff the following inferences are admissible (and all basic inferences are internalized — e.g. there are no additional inference rules, only axioms):

$$\begin{array}{ll} (\mathrm{MP}) & A, A \triangleright B \vdash B \\ (\triangleright 1) & \vdash B \triangleright (A \triangleright B) \\ (\triangleright 2) & \vdash (A \triangleright (A \triangleright B)) \triangleright (A \triangleright B) \\ (\triangleright 3) & \vdash (C \triangleright A) \triangleright ((A \triangleright B) \triangleright (C \triangleright B)) \\ (\otimes 1) & \vdash A \triangleright (B \triangleright A \otimes B) \\ (\otimes 2) & \vdash (A \otimes B) \triangleright A \\ (\otimes 3) & \vdash (A \otimes B) \triangleright B \\ (\varnothing 1) & \vdash \varnothing A \triangleright (A \triangleright B) \\ (\varnothing 2) & \vdash (A \triangleright B) \triangleright ((A \triangleright \varnothing B) \triangleright \varnothing A) \end{array}$$

It is easy to see that they make up an axiomatization of the intuitionistic propositional calculus, with $\triangleright$ acting as implication, $\otimes$ as conjunction and $\varnothing$ as negation. Hence, what we have shown is that the native inferential operators coincide (within the 'normal' environment, i.e. within standard inferential structures) with the intuitionistic ones. Thereby we reach the expected result: it is intuitionistic logic which is *the* logic of inference.

10. Multi-conclusion inference?

So far we have taken for granted: (1) that inference is a relation between finite sequences of statements and statements; and (2) that it is standard, i.e. that it complies with the Gentzenian structural rules. (It is clear that given (2), we can replace the talk of finite sequences in (1) by the talk of finite *sets*.) What if we suspend these assumptions?

Let us consider (1) first. It is well known that many logicians follow Gentzen [10] in taking inference as a relation between finite sequences of statements and *finite sequences of* statements. And though this shift might appear as an *ad hoc* means of vindicating classical logic, it is not without sophisticated defenders[14]. Here we are not going to contribute to the disputes about the naturalness or reasonableness of multi-conclusion inference;

we merely want to see what kinds of changes or modifications of our above conclusions its acceptance would effect. What comes to mind first is that in analogy to the amalgamator, the multi-conclusion inference would invite us to introduce its analogue on the right; namely the operator $\oplus$ such that

$$X \vdash Y,A,B,Z \text{ iff } X \vdash Y,A\oplus B,Z$$

Of greater interest is that it offers us more possibilities for defining the deductor: besides our (DED) and (CODED), we could consider also the more general

$$(\text{DED}^+) \quad \frac{X,A \vdash B,Y}{X \vdash A\triangleright B,Y} \qquad (\text{CODED}^+) \quad \frac{X \vdash A\triangleright B,Y}{X,A \vdash B,Y}$$

And it is well-known (see, e.g., [6]) that this makes a difference: for example, if is introduced by means of (DED$^+$) + (CODED$^+$), though not if it is introduced by means of (DED) + (CODED), it holds that

$$(\text{PL}) \quad (A\triangleright B)\triangleright A \vdash A$$

That this does not hold for a deductor introduced by (DED) + (CODED) follows from the fact that this deductor, as we have seen, yields the intuitionist implication, whereas PL amounts to Peirce's Law, notorious for being valid classically, but not intuitionistically. We will prove that it does hold for $\triangleright$ introduced in terms of (DED$^+$) + (CODED$^+$); but first we must generalize our concept of standardness from single- to multiple-conclusion inference. The structural rules with which the multiple-conclusion inference must comply in order to be standard are obvious:

$$(\text{REF}) \qquad \frac{}{X \vdash X}$$

$$(\text{EXT}) \qquad \frac{X,Y \vdash Z}{X,A,Y \vdash Z} \qquad \frac{X \vdash Y,Z}{X \vdash Y,A,Z}$$

$$(\text{CON}) \qquad \frac{X,A,A,Y \vdash Z}{X,A,Y \vdash Z} \qquad \frac{X \vdash Y,A,A,Z}{X \vdash Y,A,Z}$$

$$(\text{PERM}) \qquad \frac{X,A,B,Y \vdash Z}{X,B,A,Y \vdash Z} \qquad \frac{X \vdash Y,A,B,Z}{X \vdash Y,B,A,Z}$$

$$(\text{CUT}) \qquad \frac{X,A,Y \vdash Z \qquad U \vdash V,A,W}{X,Y,U \vdash Z,V,W}$$

[14] See, e.g., [27].

Now we can prove the promised theorem.

THEOREM 10.1. *If (DED$^+$) and (CODED$^+$) are admissible for an inferential structure, then so is (PL).*

PROOF.

1. $(A \rhd B) \rhd A, A \rhd B \vdash A$	(MP)
2. $A \vdash A$	(REF)
3. $A \vdash B, A$	from 2. by (EXT)
4. $\vdash A \rhd B, A$	from 3. by (DED$^+$)
5. $(A \rhd B) \rhd A \vdash A, A$	from 1. and 4. by (CUT)
6. $(A \rhd B) \rhd A \vdash A$	from 5. by (CON) ■

The fact that (DED$^+$) + (CODED$^+$) validate Peirce's law indicates that they, in contrast to (DED) + (CODED), would lead us to *classical* implication. And this is indeed the case. Hence we have two kinds of deductors, depending on whether or not we restrict ourselves to the single-conclusion inference. (Another one, of a relevantist kind, would be defined by means of the mere $A \vdash B$ iff $\vdash A \rhd B$.)

The situation is similar with respect to the explosion-detector ('negation'). We have at least three possibilities for capturing the intuitive idea underlying it:

$$(\mathrm{ED}^-) \quad \frac{B \vdash}{\vdash \varnothing B} \qquad (\mathrm{COED}^-) \quad \frac{\vdash \varnothing B}{B \vdash}$$

$$(\mathrm{ED}) \quad \frac{X, B \vdash}{X \vdash \varnothing B} \qquad (\mathrm{COED}) \quad \frac{X \vdash \varnothing B}{X, B \vdash}$$

$$(\mathrm{ED}^+) \quad \frac{X, B \vdash Y}{X \vdash \varnothing B, Y} \qquad (\mathrm{COED}^+) \quad \frac{X \vdash \varnothing B, Y}{X, B \vdash Y}$$

It is again clear that it is only the last version of the definition that allows us to prove the law of double negation, and hence that it introduces the *classical* negation:

THEOREM 10.2. *If (ED$^+$) and (COED$^+$) are admissible for an inferential structure, then so is $\varnothing\varnothing A \vdash A$.*

PROOF.

1. $A \vdash A$	(REF)
2. $\vdash \varnothing A, A$	from 1. by (ED$^+$)
3. $\varnothing\varnothing A \vdash \varnothing\varnothing A$	(REF)
4. $\varnothing\varnothing A, \varnothing A \vdash$	from 3. by (COED$^+$)
5. $\varnothing\varnothing A \vdash A$	from 2. and 4. by (CUT) ■

This indicates that for multi-conclusion inference, classical logic is as natural as the intuitionist one for the single-conclusion inference[15].

11. Logical operators as structural markers

Došen [5] proposed seeing logical operators as 'punctuation marks'. This is a view close to the one entertained here; only I think that it overemphasizes the syntactical function of the operators. I do not think logical operators should be seen as merely syntactic devices; hence I prefer to see them as marking certain structural features of inferential structure(s). This is very much of a piece with the view of the nature of logic put forward above: we tend to shape the frameworks of our linguistic utterances (i.e. our languages) into certain kinds of structures and we use logical vocabulary to refer to certain distinguished vertices of the structures. For example, we can say that classical as well as intuitionist conjunction refers to an *inferential infimum* of two statements: the conjunction of A and B is a statement from which both A and B are inferable and which is, moreover, the maximal statement with this property: if any other statement entails both A and B, then it must entail also their conjunction.

A theory of logical operators based on these ideas was developed by Koslow [14]. According to him, each operator maps statements on a minimum/maximum of a propositional function. Thus, for example, the conjunction of A and B is the maximal statement C such that

$$C \vdash A,$$
$$C \vdash B.$$

The maximality is understood in such a way that if there is a D satisfying the same pattern, then

$$D \vdash C.$$

Hence, from this vantage point, logical constants are devices that serve to refer to extremalities of inferential structures.

What is important is that the whole of this structure need not be explicitly articulated in the language in question, i.e. not for every vertice of the structure must there correspond a statement. The "making it explicit" that is effected by the logical operators then amounts to revealing the whole of

[15] There have also been several suggestions to admit only as much of sequent calculus into natural deduction as to allow us to handle classical logic (see [25], or [17]). I personally have suggested that we could see classical connectives as established by inferential patterns if we extend our notion of an inferential pattern (see [22]).

the structure, which is partly represented by the language in question. Let us indicate, in greater detail, what this amounts to in the case of a standard inferential structure.

An inferential structure $\langle S, \vdash \rangle$ is called *truth-preserving* if there exists a set V of truth-valuations of the set S (i.e. a subset of $\{0,1\}^S$) such that $A_1,\ldots,A_n \vdash A$ iff $v(A)=1$ for all such $v \in V$ for which $v(A_i)=1$ for $i=1,\ldots,n$. Elsewhere [21] I proved that an inferential structure is truth-preserving iff it is standard. This to say that if $\langle S, \vdash \rangle$ is standard, then it is embeddable into a Boolean algebra. Let us, conversely, assume that $\langle S, \vdash \rangle$ is embeddable into a Boolean algebra in the sense that there is a function i such that $A_1,\ldots,A_n \vdash A$ iff $i(A_1)\cap\ldots\cap i(A_n) \subseteq i(A)$. It is easy to see that this can be the case only when $\langle S, \vdash \rangle$ is standard: hence *an inferential structure is standard iff it is embeddable into a Boolean algebra.*

This indicates that there is a sense in which elements of a standard inferential structure do *implicitly* have their conjunctions, disjunctions etc. although they do not have them explicitly — if there are no expressions within the language which would express them. They do have them implicitly in the sense that they form a (proto-)structure which can naturally be extended to a structure in which these elements are present. The "naturally" can also be read as "conservatively", thus achieving the characteristic of logical operators put forward by Belnap [2], Hacking [12] and others — the addition of logical operators adds nothing substantial to the stratum of language to which it is added, it only institutes a new stratum.

Now, the view of the nature of logic put forward above is that the point of such a new stratum is in making explicit what is implicit within the old one. And natural languages appear to have the peculiar tendency to explicitate themselves in this way: what is first implicit in the behavior (making inferences) tends to find an explicit expression (in the form of a statement stating that the inference holds). This is important, for only what is explicit can be assessed, discussed and possibly also modified or rejected.

12. Substructural logics

What if we suspend the assumption of standardness; i.e. what if we consider inference as a relation which does not necessarily comply with the Gentzenian rules? Consider, for example, our definition of deductor and suppose we suspend the structural metarule (CON). Then, instead of the axiom (▷2), we will reach the weaker

$$(\triangleright 2^*) \quad \vdash A \triangleright A$$

The resulting axioms (▷1) + (▷2*) + (▷3) determine the implication of what is sometimes called the BCK logic (see, e.g., [5]), and also what constitutes the purely implicative part of Wajsberg's [36] axiomatization of Łukasiewicz's [16] three-valued calculus. As for the amalgamator, it will no longer yield classical conjunction (for (ECN1) + (ECN2) will no longer yield (AMLG)).

Suppose now that we further suspend (EXT). In this setting, (ICN) will still be equivalent to (DEAMLG), but (ECN1) + (ECN2) and (AMLG) will be independent of each other (which means that not only will (ECN1) + (ECN2) not yield (AMLG), but neither *vice versa*). However, the concept of amalgamator will coincide with what has come to be called *fussion* within the theory of substructural logics [26]:

$$\text{(IF)}\quad \frac{X\vdash A \qquad Y\vdash B}{X,Y\vdash A\otimes B} \qquad\qquad \text{(EF)}\quad \frac{X\vdash A\otimes B \qquad Y,A,B,Z\vdash C}{Y,X,Z\vdash C}$$

(The proof is straightforward.) The axioms appropriate for the deductor within this framework will then be

$$\vdash A\triangleright A;$$
$$\vdash (A\triangleright(B\triangleright C))\triangleright(B\triangleright(A\triangleright C))$$
$$\vdash (C\triangleright A)\triangleright((A\triangleright B)\triangleright(C\triangleright B));$$

Note that the set of statements together with ⊗ form a semigroup which can be represented in terms of a set of *updates* (functions on a set of 'states')[16].

In general, we can say that the basic delimitations of the inferentially native operators interact with those structural rules that are in force to produce various kinds of concrete operators. It is only within the standard and single-conclusion framework that they yield the intuitionist variety of operators; within a framework that is substructural, or that allows for other than single-conclusion inferences, things may be very different.

From the vantage point of the previous sections we can depict the tendency of natural language to 'swallow up its *meta*' as the tendency of the corresponding inferential structure to 'unfold' into a 'limit structure', which is (partly) determined by the structural rules which are in force. (The remaining part of the determination is then supplied by the possibilities of the extension, which are a matter of the general form of the rules available within the calculus — single-conclusion, multiple conclusion, . . .). Thus, a standard inferential structure within the single-conclusion calculus extends to a Heyting algebra, whereas within the multiple-conclusion calculus it extends to a Boolean algebra.

[16]See [35], Chapter 7, and [22].

Such, then, is an algebraic interpretation of the thesis that logical constants arise from the tendency of speakers to explicitate the rules implicit in their linguistic conduct: the space of algebraic structures contains certain 'attractors' which exert a pull on inferential (proto-)structures, in that the algebraic structures tend to extend themselves into them.

13. Conclusion

In so far as we take logical operators as tools of making inference explicit (which we must do if what we are after is *the logic of inference*; but what I think we should do in general), we are likely to conclude that the most natural logical operators are the intuitionist ones; hence in this sense we can say that intuitionist logic is *the* logic of inference.

However, this conclusion requires an important qualification. In particular, it presupposes that inference is, by its nature, single-conclusion and complies with Gentzenian structural rules. (I think there *are* reasons to believe this, but discussing them is not within the scope of the present technical paper.) If we reject this presupposition, the situation changes: for example, classical logic may well be seen as the logic of standard *multi-conclusion* inference.

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