EXTENSIONAL VS. INTENSIONAL LOGIC

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1 FREGE'S *BEDEUTUNG*

The German mathematician and one of the founding fathers of modern logic, Gottlob Frege (1848-1925), was the first to clearly realize that semantics has little to do with psychology, and that it could be usefully explicated in mathematical terms (see Dummett, 1973; 1981). His depsychologization of semantics followed from his depsychologization of logic. Frege understood how crucial it was for the development of logic to draw a sharp boundary separating it from psychology: to make it clear that logic is not a matter of what is going on in some person's head, in the sense that psychology is. This is because logic is concerned with what is true and consequently what follows from what — and whether something is true, or whether something follows from something else, is an objective matter independent of what is going on in the head of a particular person.

As a consequence, Frege realized that if logic must be separated from psychology, then the same is true for semantics — at least insofar as semantics underlies truth and entailment. It is clear that the truth value of a sentence depends on the meaning of the sentence: the sentence "Penguins eat fish" is true not only due to the fact that penguins do eat fish, but of course also due to the fact that the words of which it consists mean what they do in English. Hence, if meaning were a matter of what is going on in somebody's head, then truth would have to be too — hence meaning must not, in pain of the subjectivization of truth, be a psychological matter. But what, then, *is* meaning?

Frege started from the *prima facie* obvious fact that names stand for objects of the world. Unprecedentedly, he assimilated indicative sentences to names as well: he saw them as specific kinds of names denoting the two truth values: *truth* and *falsity*. The reason for this move was that he divided expressions into two sharply separated groups: into "saturated" — i.e. self-standing — and "unsaturated" — i.e. incomplete — ones. He took names and sentences as species of the former kind, whereas he took predicates as paradigmatic examples of the latter one; and he came to use the word "name" as a synonym of "saturated expression".

The reason why he identified the entities named by sentences with truth values was articulated by Frege in the form of what has later become known as the *slingshot argument*[1]. This argument itself rests on what has subsequently come to be called *the principle of compositionality* and what Frege tacitly, but unambiguously assumed[2]. The principle

[1] See Neale (2001).
[2] See Janssen (1997).

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states that the meaning of a complex expression is uniquely determined by the meanings of its parts plus the mode of their combination. This means that for every mode of combination (grammatical rule) G there must exist a function G^* such that for every e_1, ..., e_n to which G is applicable it is the case that (where $\|e\|$ denotes the meaning of e)[3]:

$$\|G(e_1, ..., e_n)\| = G^*(\|e_1\|, ..., \|e_n\|).$$

It follows that replacing a part e of a complex expression by an expression e' with the same meaning as e we cannot change the meaning of the whole complex:

$$\text{if } \|e\| = \|e'\|, \text{ then } \|G(..., e, ...)\| = \|G(..., e', ...)\|.$$

Now Frege argued that replacing names by other names of the same entities can get us from a sentence to a sentence which has very little in common with the original one; in particular that there is only one thing which inevitably persists during such a process, and this is precisely the truth value.

We can illustrate this by means of an example introduced by Alonzo Church (1956, pp. 24–25):

> [T]he denotation (in English) of "Sir Walter Scott is the author of *Waverley*" must be the same as that of "Sir Walter Scott is Sir Walter Scott," the name "the author of *Waverley*" being replaced by another which has the same denotation. Again, the sentence "Sir Walter Scott is the author of *Waverley*" must have the same denotation as the sentence "Sir Walter Scott is the man who wrote twenty-nine Waverley Novels altogether," since the name "the author of *Waverley*" is replaced by another name of the same person; the latter sentence, it is plausible to suppose, if it is not synonymous with "The number, such that Sir Walter Scott is the man who wrote that many Waverley Novels altogether, is twenty-nine," is at least so nearly so as to ensure its having the same denotation; and from this last sentence in turn, replacing the complete subject by another name of the same number, we obtain, as still having the same denotation, the sentence "The number of counties in Utah is twenty-nine."
>
> Now the two sentences, "Sir Walter Scott is the author of *Waverley*" and "The number of counties in Utah is twenty-nine," though they have the same denotation according to the preceding line of reasoning, seem actually to have very little in common. The most striking thing that they do have in common is that both are true. Elaboration of examples of this kind leads us quickly to the conclusion, as at least plausible, that all true sentences have the same denotation. And parallel examples may be used in the same way to suggest that all false sentences have the same denotation (e.g., "Sir Walter Scott is not the author of *Waverley*" must have the same denotation as "Sir Walter Scott is not Sir Walter Scott").

[3]In algebraic terms, meaning-assignment is a *homomorphism*.

Hence it seems that if we accept that a name means the thing it names, then given the principle of compositionality, two sentences are bound to have the same meaning iff they have the same truth value; and it is plausible to explicate the meanng of a sentence with its truth value.

Frege's most brilliant contribution to the explication of the concept of meaning then was the way he accounted for the meanings of predicates[4]. He called them *concepts*, as usual; but he rejected the standard view of concepts as something mental and, in effect, suggested explicating them by studying the role of those expressions which express them — i.e. predicates — within language.

What is the role of a predicate, such as "to sing"? Well, the predicate is attached to a subject, a name such as "Frege", to form a sentence, "Frege sings". Hence if we assume that the meaning of a complex expression is the result of combining the meanings of its parts (i.e. that meanings are composed in a way paralleling that in which the expressions expressing them are), then the meaning of the predicate together with the meaning of a subject, which is the object stood for by the subject, yields the meaning of a sentence, i.e. a truth value. Therefore a concept is something that together with an object yields a truth value — and this led Frege to identify concepts with functions, in the mathematical sense of the word, taking objects to truth values. In effect, this meant the identification of the meaning of an item with the semantic role of the item captured as a function in the mathematical sense of the word; and this opened the door for a mathematical treatment of semantics. Thus, we can say that Frege married semantics, which he had earlier divorced from psychology, to mathematics.

Let us reconstruct this move, which we will dub *Frege's maneuver*, in greater detail. We have a category of expressions, call it A, whose meanings we want to explicate. We see that an expression of the category A (say that of predicates) can be combined with an expression of a category B (say that of names, singular terms) to form an expression of a category C (say that of sentences). So if we take an expression a of the category A, we know that together with the expression b_1 of the category B it yields an expression c_1 of C, with b_2 it yields c_2 etc.:

$$
\begin{array}{ll}
a + b_1 = c_1 & \text{“to sing” + “Madonna” = “Madonna sings”} \\
a + b_2 = c_2 & \text{“to sing” + “Moon” = “Moon sings”} \\
\ldots & \ldots
\end{array}
$$

Hence we can see a as a means of assigning c_1 to b_1, c_2 to b_2 etc.

$$
\begin{array}{llll}
a: & b_1 \to c_1 & \text{“to sing”:} & \text{“Madonna”} \to \text{“Madonna sings”} \\
 & b_2 \to c_2 & & \text{“Moon”} \to \text{“Moon sings”} \\
 & \ldots & & \ldots
\end{array}
$$

Now suppose that we know what the meanings of both the expressions of B and of C are. Then we can transfer the whole consideration to the level of semantics:

$$
\begin{array}{llll}
\|a\|: & \|b_1\| \to \|c_1\| & \|\text{to sing}\|: & \|\text{Madonna}\| \to \|\text{Madonna sings}\| \\
 & \|b_2\| \to \|c_2\| & & \|\text{Moon}\| \to \|\text{Moon sings}\| \\
 & \ldots & & \ldots
\end{array}
$$

[4]See especially Frege (1891; 1892a).

Now if we accept that meanings of names are the objects named by them (the meaning of "Madonna", ||Madonna||, is the person Madonna, that of "Moon", ||Moon||, is the celestial body the Moon) and that the meanings of sentences are their truth values (||Madonna sings|| being *the truth*, **T**, that of ||Moon sings|| being *the falsity*, **F**), what we gain by this maneuver as the explication of the meaning of "to sing" is a function assigning truth values to individuals: **T** to those which do sing (to Madonna and others) and **F** to those which do not (to the Moon and others).

Let us further illustrate this maneuver by applying it to logical operators. Take conjunction: it joins two sentences to form a sentence. Thus it can be considered as assigning sentences to pairs of sentences. Hence on the semantic level it can be seen as assigning truth values to pairs of truth values; and hence its meaning can be identified with a function taking pairs of truth values to a truth value. Which particular function? This is revealed by inspecting the dependence of the truth values of conjoined sentences on those of their parts; and of course it is the usual well-known function assigning **T** to **T** plus **T**, and **F** to any other pair of truth values. Thus, according to Frege, meanings were either objects or functions, and the principal way of combining meanings of parts of the whole expression into the meaning of the whole was functional application.

Notice that Frege's maneuver has two substantial presuppositions. First, there is the presupposition that the meaning of a complex expression is yielded by (or 'composed of') meanings of the parts of the expressions — *viz.* the principle of compositionality. How do we know that this principle holds? Some theoreticians appear to think that it is an empirical thesis that must be verified as empirical theses are: by means of inspecting as many cases as possible. However, such a view presupposes that meanings are independently identifiable objects whose combinations can be studied in the way we study, e.g., combinations of molecules in a solution: that we can empirically verify (or falsify) the thesis that, say, the meaning of a sentence is yielded by the meaning of its subject and that of its predicate, by means of finding the meanings and finding out what they yield if they are put together. In contrast to this, we saw that for Frege the principle was rather a way of articulating what it takes to be meaning: the principle was co-constitutive of the notion of meaning in a sense analogous to that in which, say, the principle of extensionality is co-constitutive of the concept of set. And just as it makes no sense to try to find out whether sets are extensional (for this is simply part of what it takes to be a set), it makes no sense to try to find whether meaning is compositional[5].

The other presupposition of Frege's maneuver is of a different kind: it concerns the behavior of the particular expressions to which it is applied. The presupposition is that the role of the expression within language is exhausted by, or at least in some sense reducible to, its role within the kind of syntactic combination which is taken into consideration. We explicated the meanings of predicates by considering the way they combine with names into sentences; but predicates also do other things, e.g. combine with adverbials into complex predicates. We must always be sure that this is taken care of — that it is proven that it is somehow substantiated to treat some part of the functioning of an expression as representative of the whole functioning.

[5]See Peregrin (2005b). For general discussions of the principle of compositionality see also Janssen (1997; 2001) and Pelletier (2001).

Is Frege's way of explicating the concept of meaning acceptable? In fact not: what Frege called *meaning* cannot be taken as a plausible explication of the pre-theoretic notion of meaning. After all, who would want to claim that all true sentences have the same *meaning*? And Frege himself soon realized the implausibility of such an explication. Therefore he complemented his theory of meaning with what he called a theory of *sense*[6]. Every name, he claimed, has not only a meaning, but rather also a sense, which is the 'way of givenness' of the meaning. And it is then Frege's concept of *sense*, rather than his concept of *meaning*, which is to be taken as his explication of the intuitive concept of meaning.

Frege's own instructive example is that of the terms "morning star" and "evening star". As we now know, these two terms refer to one and the same celestial body, the planet Venus. Hence they share the same meaning (in Frege's sense of the word), or (in the current jargon) the same *referent*. If the referent were all that there is to meaning, it should be possible to substitute one of them for the other within any expression without changing its meaning. However, although the sentence "The morning star is the morning star" is obviously trivial, "The morning star is the evening star" does not appear to be such. The reason, Frege claimed, is that the terms differ in their senses, i.e. in the ways they present their referent: "the morning star" presents it as the most attractive body in the morning sky, whereas "the evening star" as the most attractive one in the evening sky.

Hence we have the general picture according to which the relation between a name and what the name refers to is mediated by the sense of the name:

NAME
↓
SENSE (*SINN*)
(= meaning in the intuitive sense of the word)
↓
MEANING (*BEDEUTUNG*)
(= object referred to)

One must remember, though, that despite his recognition of *Sinn*, Frege kept insisting that it is *Bedeutung* which is crucial from the viewpoint of logic. As he puts it in an unpublished text (1983, 133):

> Die Inhaltslogiker bleiben nur zu gerne beim Sinn stehen; denn, was sie Inhalt nennen, ist, wenn nicht gar Vorstellung, so doch Sinn. Sie bedenken nicht, dass es in der Logik nicht darauf ankommt, wie Gedanken aus Gedanken hervorgehen ohne Rücksicht auf den Wahrheitswert, dass, allgemeiner, der Schritt vom Sinne zur Bedeutung getan werden muss; dass die logischen Gesetze zunächst Gesetze im Reich der Bedeutungen sind und sich erst mittlebar auf denn Sinn beziehen.[7]

[6] See Frege (1892b).

[7] "The content-logicians only remain too happily with the sense, for what they call content is for them, if not mental image, then surely sense. They do not consider the fact that in logic it is not a question of how thoughts

To summarize: Frege saw as his main target *truth* (and *truth-dependence*, i.e. consequence); and took for granted that the account for it is to be *compositional*. He concluded that this can be accomplished on the level of his *meaning* (*Bedeutung*); in particular that (i) the assignment of *meanings* to expressions is compositional; (ii) the *meaning* of a sentence *is* its truth value. (The most developed logical system based on the ideas, presented in his *Grundgesetze der Arithmetik*,[8] was, however, shown to be inconsistent.)

2 CARNAP'S *EXTENSION* AND EXTENSIONAL LOGIC

Rudolf Carnap (1891-1970), the logician and logico-positivist philosopher, realized that if what we are after is meaning in the intuitive sense of the word, then we should be interested not so much in meanings in the sense of Frege, but rather in Fregean *senses*. However, as Frege did not explicate the concept of sense to Carnap's satisfaction, Carnap proposed replacing the Fregean twin concepts of *meaning* and *sense* with the concepts of *extension* and *intension*. (A laudable proposal, for it did away with Frege's misleading usage of "meaning".) And his claim is that the *extension* of a term is what the term shares with all terms that are equivalent to it; whereas its intension is what it shares with all the terms that are *logically* equivalent to it.

Of course this definition becomes non-trivial only after we give a rigorous account of the concept of equivalence on which it rests. For the basic categories of the predicate calculus this is not difficult: two individual terms t_1 and t_2 are equivalent iff

$$t_1 = t_2,$$

two n-ary predicates p_1 and p_2 are equivalent iff

$$\forall x_1 \ldots \forall x_n (p_1(x_1, \ldots, x_n) \leftrightarrow p_2(x_1, \ldots, x_n));$$

and two sentences s_1 and s_2 are equivalent iff

$$s_1 \leftrightarrow s_2.$$

This explication leads to a concept of extension almost indistinguishable from Frege's *meaning*: the extension of an individual expression being the object for which it stands, that of a predicate being the function assigning the truth value **T** to those n-tuples of objects of which the predicate is true, and that of a sentence being its truth value.

Meanwhile, what was built into the foundations of modern formal logic were extensions. The basic concept of a *semantic interpretation*, due essentially to Tarski and accepted by the mainstream of logicians,[9] was in fact that of the mapping of expressions on their extensions: names on individuals, predicates on sets of individuals or sets on their

come from thoughts without regard of truth value, that, more generally, one must make the step from senses to meanings; that the logical laws are first laws in the realm of meanings and only then mediately relate to senses."

[8]Frege (1893; 1903).

[9]The roots of the concept go back to Tarski (1936) (cf. also Etchemendy, 1990). For the definition of the concept of semantic interpretation in the modern sense, see, e.g. Mendelson (1964).

n-tuples[10] and sentences on truth values. The reasons why most logicians did not share Carnap's scruples were, roughly, two: (i) like Frege, they assumed that what is the primary aim of logic is *truth*, and that what we need to account for truth were extensions, not intensions; (ii) many of them were primarily interested in mathematics, where intensions, explicated in the Carnapian way, came to coincide with extensions. Many logicians of the second half of the twentieth century, especially those inclining to mathematics, thus came to the conclusion that there is something as *the* language of logic — the language of first-order predicate calculus (FOPC). And the semantics of this language is based on mapping expressions on their extensions.

To see this, consider FOPC as it is standardly presented. We have the logical constants $\neg$, $\wedge$, and $\forall$; the extralogical (individual, predicate, functor) constants and the (individual) variables, with the well-known syntactic rules. The semantics of the language is then based on two sets of assignments of objects to expressions: the *valuation of variables* (assignments of individuals to variables) and the *interpretation of extralogical words* (assignments of individuals to individual constants, sets of n-tuples of individuals to n-ary predicates and, n-ary functions from individuals to individuals to functor constants). Given an interpretation I and a valuation V, every individual term t is assigned a denotation $||t||_{I,V}$ based on I and V so that $||t||_{I,V} = I(t)$, if t is an individual constant, $||t||_{I,V} = V(t)$ if t is an individual variable and $||f(t_1, \ldots, t_n)||_{I,V} = I(f)(||t_1||_{I,V}, \ldots, ||t_n||_{I,V})$. An interpretation I and a valuation V then render each formula true or false, which is defined in the usual recursive way: $p(t_1, \ldots, t_n)$ is true w.r.t. (or satisfied by) I and V iff $\langle ||t_1||_{I,V}, \ldots, ||t_n||_{I,V}\rangle \in I(p)$; $\neg F$ is true w.r.t. (or satisfied by) I and V iff F is not; $F_1 \wedge F_2$ is true w.r.t. (or satisfied by) I and V iff both F_1 and F_2 are; $\forall xF$ is true w.r.t. (or satisfied by) I and V iff F is true w.r.t. (or satisfied by) I and V' for every valuation V' which differs from V at most in the value it assigns to x.

In this way, only extralogical words are taken to denote extensions; whereas the logical ones are treated as non-denoting terms. However, it is easy to do away with this difference in status by redefining an interpretation as a mapping of *all* expressions on their extensions, reducing the difference between logical and extralogical expressions to the mere fact that whereas the former ones have *fixed* interpretation, the interpretation of the latter *varies*. We may let logical constants denote the usual truth functions and we can let quantifiers denote functions mapping functions from individuals to truth values on truth values; in particular $\exists$ as denoting the function which maps f on **T** iff f maps at least one individual on **T** and $\forall$ as denoting the function which maps f on **T** iff f maps all individuals on **T**. If we do this, we can reformulate the above definition of truth as an explicit definition of denotation-assignment:

$$||p(t_1, \ldots, t_n)||_{I,V} = ||p||(||t_1||_{I,V}, \ldots, ||t_n||_{I,V})$$
$$||\neg F||_{I,V} = ||\neg||(||F||_{I,V})$$
$$||F_1 \wedge F_2||_{I,V} = ||\wedge||(||F_1||_{I,V}, ||F_2||_{I,V})$$
$$||\forall xF'||_{I,V} = ||\forall||(f),\ \text{where } f \text{ is the function which maps an individual } i \text{ on } ||F||_{I,V'}, \text{ where } V' \text{ is just like } V \text{ with the single possible exception that it maps } x \text{ on } i.$$

[10]There is usually no need to make a difference between a set and its *characteristic function*, i.e. the function which assigns **T** to elements of the set and **F** to elements of its complement.

In this way we can see semantic interpretation, i.e. the function $\|\ldots\|$, as simply an assignment of extensions.

Notice that viewed from this perspective, denotations of all expressions of the language can be seen as arising from the denotations of terms and sentences by means of Frege's maneuver. Thus, a predicate takes an n-tuple of terms into a sentence, and hence it denotes an n-ary function from individuals to truth values; a functor takes an n-tuple of terms into a term, hence it denotes an n-ary function from individuals to individuals; and a sentential operator takes one or two sentences to a sentence, hence denotes a function from truth values or their pairs to truth values. The situation is trickier w.r.t. quantifiers, which *prima facie* do *not* take predicates to sentences (which would seem to have to underlie the application of Frege's maneuver resulting into the above kind of denotations for quantifiers).

However, to straighten this it is only necessary to reassess the underlying syntax. Let us introduce a new kind of rule, the rule of *lambda-abstraction*[11], taking a formula and a variable (typically one contained free in the formula) to a unary predicate, denoting the function which maps an individual i of the universe on the extension which the formula would have if the variable denoted i. Formally, if F is a sentence and x a variable, then $\lambda x.F$ is a unary predicate whose semantics is defined in the following way:

> $\|\lambda x.F\|_{I,V}$ is the function which maps an individual i on $\|F\|_{I,V'}$, where V' is just like V with the single possible exception that it maps x on i.

Given this, we can treat a quantifier as an expression taking a unary predicate to a sentence, and take the quantified sentence $Qx.F$ as a shortcut for $Q(\lambda x.F)$ so that

$$\|\forall x.F\|_{I,V} = \|\forall(\lambda x.F)\|_{I,V} = \|\forall\|(\|\lambda x.F\|_{I,V}); \text{ and}$$
$$\|\exists x.F\|_{I,V} = \|\exists(\lambda x.F)\|_{I,V} = \|\exists\|(\|\lambda x.F\|_{I,V}).$$

This reformulation of FOPC renders all standard kinds of syntactic rules as paralleled by the same kind of combination of denotations — namely *functional application*: it is always the application of one of the denotations to the rest. (We have seen that this is the result of the fact that with the exception of terms and sentences, we can see the denotations of all other categories as furnished via Frege's maneuver.) Hence we can see the usual kind of extensional semantics as based on:

(i) letting terms denote the objects they name;

(ii) letting sentences denote truth values (which derives from (i) via the slingshot); and

(iii) letting expressions of all other categories stand for functions constructed from individuals and truth values (which derives from (i) and (ii) via Frege's maneuver).

Probably the most general language of extensional logic, based on the most consequential exploitation of Frege's maneuver, was put forward by Alonzo Church (1940) under the

[11] See Church (1956).

name of the *simple theory of types*.[12] Church's language is based on two basic categories, the category ι of individual terms and the category o of sentences. Besides this, there is a category $(\alpha\beta)$ for every two categories α and β of the language. (Thus we have categories such as (oo), $(o\iota)$, $(o(o\iota))$ etc.) The extensional semantics of the language is such that expressions of the category ι denote individuals (elements of a given universe of discourse), expressions of the category o denote truth values, and expressions of every category $(\alpha\beta)$ denote functions from the denotations of the expressions of β to those of the expressions of α. Thus, expressions of the category (oo) denote functions from truth values to truth values, those of $(o\iota)$ functions from individuals to truth values, those of $(o(o\iota))$ functions from these functions to truth values etc.

Given this semantics, the basic kind of grammatical rule of the language forms the expression $a(b)$ of the category α from an expression a of the category $(\alpha\beta)$ and an expression b of the category β; and the denotation of the resulting expression $a(b)$ results from the application of the denotation of a to that of b. To this, Church only added the above rule of lambda abstraction. The resulting language was syntactically quite rich, but wonderfully simple:

Church's Simple theory of types (Ty1)

Syntax

o and ι are types, and if α and β are types then also $(\alpha\beta)$ is a type. For each type there is an unlimited stock of constants and variables of the type.[13] If a is an expression of type $(\alpha\beta)$ and b an expression of type β, then $a(b)$ is an expression of type α; and if a is an expression of type α and x a variable of type β, then $\lambda x.a$ is an expression of type $(\alpha\beta)$.

Semantics

To each type α there corresponds a domain D_α, so that D_o is the set of the two truth values, D_ι is a universe of discourse (any given set) and $D_{(\alpha\beta)}$ is a set of functions from D_β to D_α. An interpretation maps every constant a of type α on an element of D_α; a valuation does the same for variables. If I is an interpretation and V a valuation we define the denotation assignment $||\ldots||_{I,V}$ as follows: for a constant c, $||c||_{I,V} = I(c)$; for a variable v, $||v||_{I,V} = V(v)$. Moreover, $||a(b)||_{I,V} = ||a||_{I,V}(||b||_{I,V})$ and $||\lambda x.a||_{I,V}$ is the function which maps every element i of D_β on $||a||_{I,V'}$, where V' is just like V with the single possible exception that it maps x on i.

[12]The generalization leading to this language is parallel to the one proposed by Ajdukiewicz (1935) and Bar-Hillel (1950) and which has led to what is now called *categorial grammar* (see Casadio, 1988). Bar-Hillel (1953, p. 65) expounds the idea behind this as follows: "Each sentence which is not an element is regarded as the outcome of the operation of one sub-sequence upon the remainder, which may be to its immediate right, or to its immediate left or on both sides. ('Left' and 'right' are to be understood here, as in what follows, only as the two directions of a linear order.) That sub-sequence which is regarded as operating upon the others will be called an *operator*, the others its *arguments*."

[13]The construals of languages of this kind often fluctuate between the delimitation of *specific* (though general) *language* with a maximal vocabulary so that more specific languages can be cast as its sublanguages; and the delimitation of a mere *language form* with vocabulary 'open' and to be specified only on the level of specific languages.

It is not difficult to embed the language of FOPC into this language: n-ary predicates become expressions of the category $(((o\iota)\iota)\ldots\iota)$, n-ary terms those of $(((\iota\iota)\iota)\ldots\iota)$,[14] propositional operators turn into expressions of the category (oo) or $((oo)o)$, whereas quantifiers become expressions of the category $(o(o\iota))$. The syntactical structure of Church's language is, however, incomparably richer than that of the language of FOPC.

Consider an axiomatization. We start from some axiom system of the classical propositional calculus, say

$$S_1 \to (S_2 \to S_1)$$
$$(S_1 \to (S_2 \to S_3)) \to ((S_1 \to S_2) \to (S_1 \to S_3))$$
$$(\neg S_1 \to \neg S_2) \to (S_2 \to S_1)$$
$$S_1, S_1 \to S_2 / S_2$$

(where S_1, S_2 and S_3 are expressions of the type o, $\neg$ and $\to$ are constants of the respective types (oo) and $((oo)o)$; and we write $\neg S$ and $S \to S'$ instead of $\neg(S)$ and $(\to (S))(S')$, respectively.) We add a generalization of the usual additional axioms and rule of the classical first-order predicate calculus — for every type α we have:

$$\Pi_\alpha(F) \to F(A)$$
$$\Pi_\alpha(\lambda x.(S \to F(x))) \to (S \to \Pi_\alpha(F)), \text{ where } x \text{ is not free in } S$$
$$S / \Pi_\alpha(\lambda x.S)$$

(Here S is an expression of the type o, F is an expression a type $(o\alpha)$, A is an expression of type α, x is a variable of type α, and Π_α is a constant of the type $(o(o\alpha))$.) Now it turns out that if we define $=_\alpha$ as $\lambda a.\lambda b.\Pi_o(\lambda f.(f(a) \to f(b)))$ (where f is a variable of type $(o\alpha)$), and $a$and b those of type α) all that must be added to the above axioms to obtain an axiomatization of Ty1 are the following four axiom schemes (again, for every type α and writing the equality sign in the obvious infix way):

$$(S_1 \leftrightarrow S_2) \to (S_1 =_o S_2)$$
$$\Pi_\alpha(\lambda x.(F(x) =_o G(x))) \to (F =_{(o\alpha)} G)$$
$$(\lambda x.B)(A) =_\beta B^{x \leftarrow A}$$
$$F(A) \to F(\iota_\alpha(F))$$

(where S_1, S_2, F, x are as above, ι_α a constant of type $(\alpha(o\alpha))$, G an expression of type $(o\alpha)$, B an expression of type β, $S \leftrightarrow S'$ is a shorthand for $\neg((S \to S') \to \neg(S' \to S))$, and $B^{x \leftarrow A}$ denotes the result of the replacement of x by A throughout B).

Note that as $A =_\alpha B$ amounts to A and B being intersubstitutive *salva veritate*, the first two of the axioms guarantee the 'extensionality' of the system: they guarantee that equivalent statements are always equal (hence that sentences can be treated as denoting truth values) and that functions which have the same courses-of-values are equal (and hence that they can be treated as denoting the very courses-of-values, which is the essence

[14] In this way, we take an n-ary predicate (functor) to take not an n-tuple of terms to a sentence (term), but rather a term to an (n-1)-ary predicate (functor); hence the combination of the n-ary predicate (functor) with n individuals comes out as n subsequent applications. This purely formal maneuver makes it possible to make do with merely unary predicates (functors).

of Frege's maneuver). The third axiom, called *lambda-conversion*, guarantees the proper semantic working of the λ-operator;[15] whereas the last axiom is a sort of an axiom of choice that guarantees that whenever a set is not empty, we can name an element of the set (which implies that whenever there is a true existential statement, we have the name of a corresponding witness).[16]

Completeness of a slightly different version of this axiom system was proved by Henkin (1950). However, the completeness proof is possible only because we defined the semantics so that $D_{(\alpha\beta)}$ is *a* set of functions from D_β to D_α; once we replace "a set of functions" by "*the* set of *all* functions", we get a semantics which is not axiomatizable (in consequence of the Gödel theorem). The former kind of semantics is usually called the *Henkin semantics*, whereas the latter is called *standard semantics*.[17] Hence the above axiomatization of Ty1 are complete w.r.t. the Henkin semantics of the language, not w.r.t. the standard one.

Thanks to the expressive richness of the language we can also take all the usual logical operators as *defined symbols*; taking only = (for every type) as primitive:

$$
\begin{aligned}
&\mathbf{T} \equiv_{Def.} (\lambda y.y =_{(oo)} \lambda y.y)\\
&\mathbf{F} \equiv_{Def.} (\lambda y.y =_{(oo)} \lambda y.\mathbf{T})\\
&\Pi_\alpha \equiv_{Def.} \lambda g.(g =_{(o\alpha)} \lambda a.\mathbf{T})\\
&\wedge \equiv_{Def.} \lambda y.\lambda z.(\lambda h.h(\mathbf{T},\mathbf{T}) =_o \lambda h.h(y,z))\\
&\rightarrow \equiv_{Def.} \lambda y.\lambda z.(y =_o y \wedge z)\\
&\neg \equiv_{Def.} \lambda y.(y =_o \mathbf{F})\\
&\vee \equiv_{Def.} \lambda y.\lambda z.\neg(\neg y \wedge \neg z)
\end{aligned}
$$

(where y, z, a, g, h are variables of the respective types $o, o, \alpha, (oo)$ and $((oo)o)$ and we write $h(x,y)$ instead of $(h(x))(y)$ and use the usual infix notation for logical operators).

Andrews (1986) has shown that then we can make do with the following axioms and rule:

$$
\begin{aligned}
&H(\mathbf{T}) \wedge H(\mathbf{F}) =_o \Pi_o(\lambda y.H(y))\\
&B =_\beta B' \rightarrow (I(B) =_\alpha I(B'))\\
&(I =_{(\alpha\beta)} I') \rightarrow \Pi_\beta(\lambda x.(I(x) =_\alpha I'(x)))\\
&(\lambda x.B)(A) =_\beta B^{x\leftarrow A}\\
&\iota_\alpha(\lambda x.(x =_\alpha A)) =_\alpha A\\
&A =_\alpha A', B/B^{A\leftarrow A'}
\end{aligned}
$$

Here the newly introduced letters H, B', I and I' stand for expressions of the respective types $(oo), \beta, (\alpha\beta), (\alpha\beta)$.

To summarize: we can see the semantics of first-order predicate calculus (which underlies also common model theory) as a matter of connecting expressions to their extensions

[15] As Andrews (1986) duly points out, this axiom can be seen as a disguised comprehension principle: it involves that to every open formula there corresponds a function, especially that for every formula of type o there is the function which maps all n-tuples satisfying the formula on **T** and all others on **F**.

[16] The ι operator is thus reminiscent both of the 'iota inversum' operator of Russell, and of the ε operator of Hilbert.

[17] For the concept of the Henkin semantics, see, e.g., Andrews (1986) or Shapiro (1991).

such that the extension of a whole is always computed from the extensions of parts via a functional application; and we can see Ty1 as the ultimate generalization of this idea.

3 FAILURES OF EXTENSIONALITY AND CARNAP'S *INTENSION*

As we saw, Carnap pointed out that if what we are after is *meaning* in the intuitive sense, then extensions would not do. We saw that in fact this is rather obvious, for probably nobody would be willing to admit that a truth value is what a sentence *means* and that consequently all true sentences (as well as all the false ones) are synonymous. However, Carnap used more sophisticated counterexamples, mostly concerning predicates. Thus, for instance, he claimed (borrowing from Aristotle's classification) that though the predicates "human" and "featherless biped" are co-extensional (there is, as a matter of fact, no animal species which has no feathers and at the same time has two legs, save us, humans), they are clearly not synonymous. (We all know that a non-human featherless biped is surely conceivable — e.g. a hobbit!).[18]

This means that insofar as a logician wants to explicate the concept of meaning, she must not stay on the level of extensions and must follow Carnap to that of intensions. However, should a logician struggle to explicate meaning at all? Or should she care only about truth, and hence, as Frege maintained, extension? We have seen that Frege endorsed extensions in view of the fact that they are enough to provide for a compositional account for truth; in particular that (i) the assignment of extensions to expressions is compositional; (ii) the assignment of extensions to sentences coincides with the assignment of truth values.

It is important to realize that the Carnapian considerations challenge not only the extensions' capability of explicating the intuitive concept of meaning, but rather also the Fregean way of accounting for truth, by challenging the assumption (i). Consider the sentence

(1) One need not know that all featherless bipeds are human.

It is clearly true: one who has not studied much zoology need not know such thing. However, if extensions are compositional, then we are free to replace the name of an extension by another name of the same extension within the sentence without thereby changing the truth value of the sentence. Hence it seems that also the sentence

(2) One need not know that all humans are human,

which arises out of replacing "featherless biped" by the co-extensional "human", must also be true. But how could anybody sensibly fail to know that humans are humans?

Or consider a sentence of the shape

(3) One plus one is necessarily two.

Such sentences have traditionally been considered as resulting from the application of the operator of necessity to a sentence; hence its extension, i.e. its truth value should be

[18] See Carnap (1947, §5).

yielded by a combination of the extension of "necessarily" and the extension (truth-value) of "one plus one is two". It follows that replacing the sentence "one plus one is two" by any other sentence with the same truth value should not change the truth value of (3). However, it is easy to see that if we replace "one plus one is two" by a sentence which is true only contingently, perhaps "Prague is the capital of Czechia", then the truth value *will* change from **T** to **F**.

Hence it is clear that extensions *in general* are not compositional. (What would be Frege's response to this? Did he not see this? He surely did, but he was convinced we could restrict ourselves to only that part of language where the compositionality of extensions does hold — that this is the 'logically relevant core' of language.) Thus a non-extensional logic is needed not only when we are engaged within the philosophical enterprise of explicating the concept of meaning, but also when we want to logically master certain, 'non-extensional' contexts.

All in all, it seems that if what we want is either to explicate the concept of meaning, or to account for non-extensional contexts, we should follow Carnap and turn our attention to intensions. However, what is an intension? Is there a possibility of explicating it as explicitly as extension and to establish an 'intensional logic'?

We saw that Carnap considered two expressions co-extensional if they were equivalent in the precise sense defined in the beginning of the previous section. This yielded him the explication of extensions. Now turning his attention to intensions, he concluded that they can be approached in an analogous way: he proposed to consider two expressions co-intensional if they are *logically* equivalent, i.e. if their equivalence is not simply true, but *logically* true (their equivalence follows by nothing else than the laws of logic). Thus while "the morning star" and "the evening star" are co-extensional, for

(4) The morning star is the evening star

is true, they are not co-intensional, for (4) is not *logically* true. Similarly "featherless biped" and "human" are — for all we know — co-extensional, for

(5) An individual is a featherless biped iff it is human

is true; but they are not co-intensional, for (5) is not *logically* true.

While this meant an advance on the way to explicating co-intensionality, it was not, however, directly on the way to explicating *intension*. But Carnap indicated also a more promising route to the latter goal: he noticed that two sentences are logically equivalent iff they are true w.r.t. the same states-of-affairs. Thus, Carnap (1947), introduced the concept of *state-description*. A state description is a set of atomic sentences containing for every atomic sentence either it, or its negation. Then Carnap assigned to every sentence what he called its *range*: the set of all state-descriptions in which it is true.

Intuitively, state-descriptions represent conceivable states of our world (as Carnap himself puts it, *ibid*., p. 9, they represent "Leibniz's possible worlds or Wittgenstein's possible states of affairs"); formally each of them uniquely determines a maximal consistent set of sentences. Carnap's observation indicated that the intension of a sentence could perhaps be explicated precisely as its range: as the set of all those state-descriptions (and consequently 'possible worlds' — whatever these may be) in which the sentence is true. And precisely this has later become the point of departure of modal and intensional semantics.

4 MODAL LOGIC

In fact logicians did ponder sentences claiming that something is possible or that it is necessary from the very beginning of the enterprise of logic, i.e. since the time of Aristotle. And besides the majority of post-Fregean logicians, who developed the wonderfully simple and transparent extensional logic outlined by Frege (and also Boole and others), there was always a minority of those who thought about ways of integrating even the non-extensional contexts into logic.

In 1932 Lewis and Langford published their *Survey of Symbolic Logic*, in the Appendix of which C. I. Lewis presented five axiomatic systems of modal logic (thereafter known as S1–S5). The crucial logical connective which appeared in them was the so-called *strict implication* ($\prec$); however, the modern modal logicians came to the conclusion that as this is one of the triad of mutually interdefinable modal operators, the other members of which being *necessity* ($\Box$) and *possibility* ($\Diamond$), we are free to take any one of the three as the primitive one and mostly settled for $\Box$. ($\Diamond S$ then can be defined as $\neg\Box\neg S$, whereas $S_1 \prec S_2$ as $\Box(S_1 \rightarrow S_2)$). Therefore most modern axiomatic systems of modal logic are based on the necessity operator (usually called the *box*; the possibility operator being called the *diamond*).[19] It is, however, good to realize that there is a respect in which Lewis' strict implication is more important than its two relatives: while the necessity and the possibility operators do not correspond to anything terribly important in ordinary discourse[20], the strict implication seems to correspond to the all-important contrafactual conditional of natural language.

It was clear that the language of modal logic did not allow for an extensional interpretation. $\Box$ was syntactically on a par with $\neg$, hence within the extensional framework it would have to denote a unary truth function. Assume that such a function exists; call it $f_\Box$. And assume that within the language we are interested in there is a sentence, say "1=1", which is necessarily true, and another sentence, perhaps "Prague is the capital of Czechia", which is true, but not necessarily (if there were no such two sentences, the situation would be trivial). Then it must be the case that

$$\begin{aligned}\mathbf{T} &= \|\Box(1=1)\| = f_\Box(\|1=1\|) = f_\Box(\mathbf{T}) = \\ &= f_\Box(\|\text{Prague is the capital of Czechia}\|) = \|\Box(\text{Prague is the capital of Czechia})\| = \\ &= \mathbf{F}.\end{aligned}$$

Hence, in pain of contradiction, no such function exists.

This means that if modal logic is to be interpreted, we need a semantics which is not extensional. The first attempts to build a feasible semantics were presented in the fifties;

[19] For a detailed exposition of modal calculi, see Hughes & Cresswell (1968), Chellas (1980) Gamut (1991), or Blackburn et al. (2000).

[20] This was argued for by Quine. He pointed out (1992, p. 73) that when, in ordinary discourse, we apply the adverb "necessarily" to a sentence, we usually do not mean to express anything of the kind studied by modal logic, but rather either want to express that the sentence is "presumed acceptable to our interlocutor and stated only as a step toward the consideration of moot ones" or want to "identify something that follows from generalities already expounded, as over against new conjectures or hypotheses". Thus, Quine concludes that "expression [of necessity] serves a purpose in daily discourse, but a shallow one."

they culminated in the work of Saul Kripke, who is nowadays usually considered as *the* author of such a semantics.[21] Let us consider his proposals in detail.

Kripke came to the conclusion that we must let sentences denote not truth values, but rather subsets of a given set. He called elements of the underlying set *possible worlds*, which made his proposal (treacherously) easy to grasp: each sentence is taken to denote the set of those possible worlds in which it is true. This further lets us explicate necessity as 'truth in every possible world' and possibility as 'truth in at least one possible world', which is again very plausible. Formally, given a set W of possible worlds, we can define $f_\Box$ as follows:

$$\begin{aligned} f_\Box(x) &= W \text{ iff } x = W \\ &= \emptyset \text{ otherwise.} \end{aligned}$$

This semantics yields a rather plausible class of tautologies, like $\Box S \to S$ ('if something is necessarily the case, then it is the case'), $\Box S \to \Box\Box S$ ('if something is necessarily the case, then it is necessary that it is necessarily the case') etc.

Aside of the story about 'possible worlds', there is also a purely algebraic way to understand Kripke's proposal. The ordinary (extensional) logical connectives make the set of denotations of sentences into a Boolean algebra (with conjunction acting as the meet, disjunction as the join and negation as the complement). As there is no need for a more complicated Boolean algebra than the simplest, two-element one, we make do with the two truth values. However, the introduction of the modal operators blocks this simplest possibility; so the solution is to settle for less trivial Boolean algebras, and as every Boolean algebra can be represented by the algebra of subsets of a set, we have the semantics outlined above.[22]

In fact, Kripke proposed a more sophisticated semantics[23], in which the set of possible worlds was supplemented by a binary relation, which is called the *accessibility relation*. The idea is that what would be relevant for the necessary truth of S w.r.t. a world w is not the truth of S w.r.t. all worlds whatsoever, but rather only its truth w.r.t. those which are *accessible* from w. Thus, if we are, for example, trying to explicate *physical* necessity, then we might, considering necessity in a world w, want to disregard those worlds in which there are different physical laws (and hence we make the accessibility relation into the relation of *sharing the same physical laws*).

It has turned out that by fine-tuning the accessibility relation we can develop sound and complete semantics for a rich variety of axiomatically defined modal logics. In particular, it has turned out that many axioms of modal logic correspond to simple properties of the accessibility relation. Thus, for example, $\Box S \to S$ holds if and only if the relation is reflexive; and similarly for many other axiom-candidates. Investigations into this kind of correspondence has generated what is now called the *correspondence theory*.

There is a further way of understanding the step from the extensional to the intensional, possible-world semantics. We can imagine that we simply change the extensional semantics so as to admit that the 'actual' distribution of truth values among sentences is not the only one possible — hence that we have a whole set of acceptable truth valuations.

[21] See Copeland (2002) for a detailed overview of the emergence of possible-worlds-semantics.
[22] Cf. Peregrin (2005a).
[23] See Kripke (1963; 1965).

Each such valuation corresponds, on the one hand, to a Kripke possible world, while, on the other, can be seen as amounting to a Carnap *state-description*, as a means of pointing out certain sentences. In this way, Kripke's proposal come to connect with the one of Carnap.[24]

5 MODAL PREDICATE LOGIC

There is a limited parallel between modal propositional logic and extensional predicate logic. We may compare possible worlds with individuals and modalities with quantifiers: indeed in the simplest case,

> necessarily *S*

means

> for (in) every world, *S*

whereas

> possibly *S*

means

> for (in) at least one world, *S* .

In the general case the situation is more complicated, but still we can say that

> necessarily *S*

means

> for (in) every accessible world, *S*

whereas

> possibly *S*

means

> for (in) at least one accessible world, *S* .

This opens the way for a reduction of some modal logics to the extensional first-order predicate logic.[25] (For other systems of modal logic this may not be possible because

[24]An approach to the semantics of modal logic alternative to that of Kripke and more congenial to that of Carnap was fostered by Hintikka (1969). Its basic concept of *model set* is quite close to Carnap's *state description*.

[25]As we have already pointed out, $\Box S \rightarrow S$, for instance, is valid within a modal logic iff the underlying accessibility relation is reflexive, i.e. iff, denoting the relation as R, $\forall w.w\mathrm{R}w$.

some modal formulas correspond to such properties of the accessibility relation that are not first-order definable[26].)

However, once we have a propositional logic of possibility and necessity, we would want to extend it to a predicate logic. On first sight, this should be simply a matter of adding the necessity operator to the standard predicate calculus; but in fact the situation is somewhat trickier. Whereas in extensional predicate calculus we have the universe of individuals and in modal propositional calculi we have the universe of possible worlds, which can be seen as, in a sense, replacing it, in modal predicate logic we have *both*, and we face the problem of their mutual relationship.

This is indeed a problem, though *prima facie* the situation might seem to be clear: it appears that the universe of individuals must be relativized to possible worlds, that there should be a separate universe for each world. However, the situation is not so simple. First, this would imply considerable complications for the formal apparatus. The point is that whereas the formula $\Box\forall xS$ ("for every world and every its individual, S") would continue to make sense, $\forall x\Box S$ would become senseless, for it would require us to quantify over individuals which do not belong to any possible world, which would make no sense.

Besides this, there is a less formal problem. One of the basic points of modal logic is the analysis of contrafactual locutions, which in natural language typically have the form "If [it were the case that] ..., then [it would be the case that] ...". This means that one of the prototypical kinds of sentences which we might hope to capture by means of modal predicate calculus is that exemplified by

(6) If I were the president of Russia, I would make St. Petersburg the capital.

And it seems that this requires that *the same* individual be in *more than one* possible world: namely that there is a possible world in which I, the same person which exists in the actual world, am the president of Russia.

There are a number of responses to this problem. Thus, D. Lewis (1968; 1986), for example, wanted to save the idea of the world-specific universes by postulating a *counterpart relation* among individuals of the universes of different worlds. Hence though I do exist only in the actual world, in other worlds there are my counterparts which are, for the purposes of semantic analysis, indistinguishable from me.

Tichý (1971), on the other hand, accepted that individuals are prior to possible worlds: which made him reduce individuals to bare 'property-hangers'. That is, the individuals outside of possible worlds do not have any non-trivial properties (though they are 'numerically' distinct from each other), and they acquire such properties only within the context of possible worlds: and possible worlds then emerge from a distribution of some basic properties among the transcendental individuals.

However, also this solution has consequences which appear unwanted. It implies, for example, that there cannot be a possible world in which I do not exist — in some worlds I may be a stone or an ash-tray (or perhaps even a summer breeze?), but there is for me no way of not being there at all. It also implies that there are worlds which do not differ in

[26]Thus, for example, to $\Box\Diamond S \rightarrow \Diamond\Box S$ there arguably corresponds no property of the accessibility relation which would be expressible within the language of FOPC (see Goldblatt, 1975). Another example is $\Box(\Box S \rightarrow S) \rightarrow \Box S$ (see Boolos, 1979, p. 82).

any humanly recognizable way, but which *are* different in that, for example, the property-hanger underlying a person A in one of them underlies a person B in the other and *vice versa*.

These two extreme approaches may be partly reconciled by means of differentiating two senses of "exists": there is a broader sense, in which every individual which can be found in any possible world exists (*simpliciter*); and there is a narrower sense in which existence is relativized to a possible world and in which only what can be found in a world exists in that world. Some individuals which exist in the broader sense may not exist in the narrower sense in a given possible world. Existence-in-a-world then can be conceived of as a *property*, a property which in every possible world is instantiated by those individuals which exist-in-the-world.

The way we choose to construe the relationship between the universe of individuals and that of possible worlds also bears on the validity of the so-called Barcan formula. This formula codifies the interchangeability of the quantifications over the two universes:

$$\forall x \Box F \leftrightarrow \Box \forall x F.$$

If the universes are treated as independent (in the sense that we take all individuals to exist in every possible world — so that the quantification over individuals is always taken to be over the whole universe), the formula is unproblematically true. On the other hand, if the individuals are relativized to possible worlds (so that the quantification over individuals within a possible world is only over the individuals which exist in that possible world, which need not be all individuals), then this formula, if not deemed utterly meaningless (see above), may hold only for such interpretations in which the universe of individuals of every possible world would be the same as that of every possible world accessible from it.

It seems that from the intuitive viewpoint, it would be natural to have both the possibility of the same individual occurring in more than one world and the possibility of an individual present in a world being utterly absent from another world. However, this would obviously lead to an apparatus much more complicated and much less elegant than the above two.[27]

6 MONTAGUE'S 'LOCALLY' INTENSIONAL LOGIC

The most famous system of modal predicate logic was presented by Richard Montague (1970a; 1970b; 1973). He called the system *intensional logic* and this term has been almost universally accepted, so that the usage of the term "modal" is now normally restricted to propositional logic. The basic idea was that of *explicating intensions generally as functions from possible worlds to extensions*. This was compatible with how we could see the relationship between the extension of a sentence and its Kripkean intension: the

[27] See Gamut (1991, §3.3) for a further discussion of the choice between an absolute universe and relative universes. For discussions of further aspects of the 'metaphysics of possible worlds' see Loux (1979) or Divers (2002).

intension of a sentence could be seen as a function mapping each world on the truth value of the sentence for that world.

This means that while an intension of a sentence is a function from possible worlds to truth values, that of an individual term is a function from possible worlds to individuals and that of a predicate is a function from possible worlds to classes of individuals or classes of *n*-tuples of individuals. Thus, the intension of “the president of the USA” is a function mapping every possible world on its president of the USA (if any); whereas the intension of “featherless biped” is a function mapping every world on the class of its featherless bipeds. In this way, Carnap’s idea of ranges gets generalized to expressions of all categories.

However, this elegant solution brings with it a grave problem. We saw that within extensional semantics the uniform way of combining denotations of parts into the denotation of a whole was functional application; but if we explicate intensions as Montague did, this is no longer possible. While we can apply the extension of “featherless biped” to the extension of “the president of the USA” to yield us the extension of “The president of the USA is a featherless biped” (for the former extension is a function from individuals to truth values and the latter is an individual), we cannot do the same with the respective intensions — the intension of “featherless biped” is no longer the kind of function which would be applicable to the intension of “the president of the USA”.

What we can do is to take the values (i.e. extensions) of these intensions for a particular world and let them yield us the extension of “The president of the USA is a featherless biped” for the possible world. And we can do this for any possible world. Hence as the intension of the sentence is uniquely determined by its truth values w.r.t. all possible worlds, the intension can be obtained by obtaining the extensions for every possible worlds. This was Montague’s strategy: he gave the rules for computing extensions assuming that their totality yields us also intensions.

Obviously, the situation is not so simple that we could always merely take extensions of components and use them to yield us the extension of the corresponding compound — if so, then the difference between extensional and intensional logic would be rather trivial. The *raison d’être* of intensional logic, we saw, was the fact that in some cases we may need *more* than just the extension of a component to get the extension (not to speak about intension) of the compound.

Montague’s solution was that (i) each expression of his logic had both an extension and an intension; (ii) in extensional contexts expressions continued to denote their extensions; and (iii) there was a mechanism which would take care of intensional contexts in that it would allow, in effect, an expression to ‘exceptionally’ denote its intension instead of the extension. Thus, Montague’s was what we could call a *locally* intensional logic — intension enters the scene only where there is no way of making do with extension. The mechanism with which Montague accomplished this was realized by an operator, namely $^\wedge$. Its role was such that if we denote the extension of an expression e as $||e||_E$ whereas its intension as $||e||_I$, we can write

$$||^\wedge e||_E = ||e||_I.$$

Thus, as “necessarily”, as we saw, constitutes an intensional (i.e. non-extensional) context, the formula corresponding to the natural language *necessarily S* will not be $\Box S$, but rather $\Box{}^\wedge S$. And while the extension of S is a truth value, that of ${}^\wedge S$ equals the intension of S and hence is a set of possible worlds.

Let us describe Montague's system in greater detail.[28] However, we will present it in a slightly simplified form and also using a different notation than Montague, in order to stress the continuity with Church's Ty1 described above[29].

Montague's intensional logic (MIL)

Syntax

ι and o are basic types, and if α and β are types then also $(\alpha\beta)$ is a type. Moreover, if α is a type, then also $(\alpha\omega)$ is a type. For each type we have an unlimited stock of constants and variables. If a is an expression of type $(\alpha\beta)$ and b an expression of type β, then $a(b)$ is an expression of the type α; and if a is an expression of type α and x a variable of type β, then $\lambda x.a$ is an expression of type $(\alpha\beta)$. Moreover, if a is an expression of type α, then ${}^\wedge a$ is an expression of type $(\alpha\omega)$; and if a is an expression of type $(\alpha\omega)$, then ${}^\vee a$ is an expression of type α.

Semantics

To each type α there correspond two domains S_α and D_α, where S_α is identical to $D_{(\alpha\omega)}$. D_o is the set of the two truth values, D_ι is a universe of discourse (any given set); $D_{(\alpha\beta)}$ is a set of functions from D_β to D_α, and $D_{(\alpha\omega)}$ is a set of functions from a set of 'possible worlds' (any given set) to D_α. An interpretation maps every constant a of type α on an element of S_α; a valuation maps every variable x of type α on an element of D_α. If I is an interpretation and V a valuation, then for every expression a of the language and every possible world w we define the extension $||a||^w_{I,V}$ of a in w, thereby defining the intension $||a||_{I,V}$ of a, as the function mapping every w on $||a||^w_{I,V}$. For a constant c, $||c||^w_{I,V}$ $= (I(c))(w)$; for a variable v, $||v||^w_{I,V} = V(v)$. $||a(b)||^w_{I,V} = ||a||^w_{I,V}(||b||^w_{I,V})$ and $||\lambda x.a||^w_{I,V}$ is the function which maps every element i of D_β on $||a||^w_{I,V'}$, where V' is just like V with the single possible exception that it maps x on i. Moreover, $||{}^\wedge a||^w_{I,V}$ is the function which maps every possible world w' on $||a||^{w'}_{I,V}$; whereas $||{}^\vee a||^w_{I,V}$ is $||a||^w_{I,V}(w)$.

As an example, consider the sentence

(7) Madonna sings

In terms of Ty1 (or, for that matter, FOPC), it might be analyzed simply as

(7′) **sing**(**Madonna**)

where **sing** is an expression of the type $(o\iota)$ and **Madonna** of the type ι. Now passing over to MIL, nothing needs to change; save for the fact that each of the expressions as well as

[28] See Mongague (1974); and also Gallin (1975), Partee (1976) and Gamut (1991, Chapter 6).

[29] Montague used the letters e, t and s instead of ι, o and ω, and he also wrote $\langle b, a\rangle$ instead of $(\alpha\beta)$. Hence, for example, his equivalent of $(o(o\iota))$ was $\langle\langle e, t\rangle, t\rangle$.

the whole sentence will now have also also an intension; the intension of the sentence being denoted by $^\wedge$(**sing**(**Madonna**)).

To illustrate some idiosyncrasies of Montague's approach, consider a slightly more sophisticated example:

(8) John finds a unicorn

Within FOPC, the most straightforward analysis would be

(8′) $\exists x\,(\mathbf{find}(\mathbf{John}, x) \;\wedge\; \mathbf{unicorn}(x))$.

Switching to Ty1, we need to make the binary predicate **find** into an expression of the type $((o\iota)\iota)$ (taking a term to a unary predicate):

(8^2) $\exists x\,(((\mathbf{find}(x))(\mathbf{John})) \;\wedge\; \mathbf{unicorn}(x))$.

Using the mechanism of lambda-abstraction, this can be further transformed to

(8^3) $(\lambda y.\exists x(((\mathbf{find}(x))(y)) \wedge \mathbf{unicorn}(x)))(\mathbf{John})$.

Now if we define (where r is a variable of the type $(o(o\iota))$)

$$\mathbf{find^*} \equiv_{Def.} \lambda r.\lambda y.r(\lambda x.(\mathbf{find}(x))(y)),$$

this can be further turned into

(8^4) $(\mathbf{find^*}((\lambda p.\exists x(p(x) \wedge \mathbf{unicorn}(x)))))(\mathbf{John})$

Here **John** is a constant of the type ι, x is a variable of the same type, p is a variable of the type $(o\iota)$, and **unicorn** and **find*** are constants of the types $(o\iota)$ and $((o\iota)(o(o\iota)))$, respectively. (Note that the step from **find** to **find*** was a purely technical one, relying on the fact that an individual, or any other object, can be identified with the class of all classes to which it belongs; hence though the type of the counterpart of "find" should be intuitively $((o\iota)\iota)$, **find*** is of the type $((o\iota)(o(o\iota)))$.)

Now consider a similar sentence

(9) John seeks a unicorn,

which, however, differs from (8) in the crucial respect that the object position of "seek", unlike that of "find", constitutes an intensional context. (Whereas you cannot find a unicorn without there being one, you can perfectly well seek a unicorn even when no unicorns exist.) Thus the counterpart of "seek", unlike that of "find", should be intuitively of the type $((o\iota)(\iota\omega))$, but due to the maneuver explained above we have **seek*** of the type $((o\iota)(o(o(\iota\omega))))$. Hence we must use, instead of p, a variable q of the type $o(\iota\omega)$ and, consequently, 'intensionalize' its argument:

(9′) $(\mathbf{seek^*}((\lambda q.\exists x((q)(^\wedge x) \wedge \mathbf{unicorn}(x)))))(\mathbf{John})$.

Montague's own analysis only differs from this one in two minor points: first, he lets *all* arguments of *all* predicative expressions undergo intensionalization (which forces different types of the predicative expressions); and, second, he uses variables which have only extensions and no intensions. This results into the formula[30]

$$(9^2)\quad (\mathbf{seek*}(\lambda q.\exists u(\mathbf{unicorn}(^\wedge x) \wedge (^\vee q)(^\wedge x))))(^\wedge\mathbf{John}),$$

where q is a variable of the type $((o(\iota\omega))\omega)$, and **unicorn** and **seek*** are constants of the types $(o(\iota\omega))$ and $((o(\iota\omega))(o((o(\iota\omega))\omega)))$, respectively. It is clear that only to comprehend the type of the verb is anything but easy, and to decipher the whole formula is even harder.

An axiomatization of Montague's intensional logic was presented by Gallin (1975). It turns out that there are only two axioms which must be added to an axiomatization of Ty1, namely

$$\Box(^\vee x = {}^\vee y) \rightarrow (x = y);$$
$$^{\vee\wedge}x = x,$$

where

$$\Box x \equiv_{Def.} (^\wedge x = {}^\wedge\mathbf{T}).$$

7 'GLOBALLY' INTENSIONAL LOGIC

Thus Montague still declares extensions as the basic semantic values, and takes intensions to be relevant only in that they can temporarily assume the place of extensions. Therefore he needs the operator $^\wedge$, which is, however, not unproblematical in that it is, in contrast to usual logical operators, *not* defined compositionally. In particular, there is no function which would take us from the semantic value (extension) of e to that of $^\wedge e$. This is understandable in that insofar as Montague sees extensions as the basic semantic values, intensionality cannot but be a failure of compositionality. Note however that though $^\wedge$ is generally applicable, there is no way of defining it generally: going from $||e||$ to $||^\wedge e||$ is not a matter of a rule or of an algorithm, but rather of the advance knowledge of $||^\wedge e||$.

We can say that facing the dilemma of either taking intensions at face value (i.e. regarding them directly as denotations of sentences) or saving the Fregean paradigm (according to which one constituent of every complex expression denotes a function applicable to the denotations of the others), Montague voted for the second alternative and hence treated expressions as denoting extensions, letting intensions do their job only via the mediation of the operator $^\wedge$. However, it seems that to embrace the first alternative may be in some respects better. We may let expressions denote directly intensions (thereby achieving what can be called *globally* intensional logic) and modify the Fregean paradigm.

Let f be a function from D_ω to $D_{(\alpha\beta)}$ and let g be that from D_ω to D_β. Let us use the term *intensional application* of f to g for the operation which produces the function h from D_ω to D_α such that for every $x \in D_\omega, h(x) = (f(x))(g(x))$. It is easy to see that if we change Montague's semantics in such a way that every expression comes to denote its intension and every application of a denotation to other denotations becomes an *intensional* application, everything will work as smoothly as before.

[30] See Montague (1973).

Moreover, it takes only a minor enhancement of the language of MIL to be able to articulate intensional application explicitly. If we add variables ranging over possible worlds, then obviously the result of the intensional application of the denotation of an expression A of the type $((\alpha\beta)\omega)$ to an expression B of the type $(\beta\omega)$ will be denoted by $\lambda w.A(w)(B(w))$. Thus, the intension of (9) will be denoted by (where for the sake of comprehensibility we write the argument w as a subscript):

(9[3]) $\lambda w.(\mathbf{seek*}_w(\lambda w.\lambda q.\exists x(\mathbf{unicorn}_w(x) \wedge q(x))))(\mathbf{John})$

where the respective types of q and **unicorn** have been changed to $((o\iota)\omega)$ and $(o\iota)$, whereas that of **seek*** to $(((o\iota)((o(o\iota)\omega)))\omega)$.

In what respect may a globally intensional logic be considered as preferable to the locally intensional one? First, there is some slight simplification both of the types corresponding to the common expressions of natural language and of the formulas which emerge from their analyses. Second, we get rid of Montague's problematic operators $^\wedge$ and $^\vee$. And third, we can take the theoretical relation of denotation as a reasonable explication of the pre-theoretical relation of meaning, for now it relates expressions to their *intensions*, which, unlike extensions, are capable of serving as explicata of meanings (though not as perfect ones[31]).

A simple way to achieve globally intensional logic is to elevate possible worlds to a fully-fledged type. Hence we have, in addition to the basic types o and ι, also the type ω of possible worlds; and of course all the types which can be obtained from all the three. And we assume that each expression is mapped simply on its intension (a function from possible worlds to the corresponding extensions).

This logic, within the context of post-Montagovian research, was presented by Gallin (1975) under the name of *two-sorted type theory* (or Ty2, as a generalization of Church's system with a second domain, besides that of individuals).[32] However, it should not escape our attention that an intensional logic based on this approach was presented independently by Tichý (1975; 1978a; 1978b).

Ty2 or Tichý's intensional logic

Syntax

ι, o and ω are basic types, and if α and β are types then also $(\alpha\beta)$ is a type. For each type we have an unlimited stock of constants and variables. If a is an expression of type $(\alpha\beta)$ and b an expression of type β, then $a(b)$ is an expression of type α; and if a is an expression of type α and x a variable of type β, then $\lambda x.a$ is an expression of type $(\alpha\beta)$.

[31] See the last section.
[32] For a thorough discussion of the relationship between MIL and Ty2, see Zimmerman (1989).

Semantics

To each type α there corresponds the domain D_α. D_o is the set of the two truth values, D_ι is a given universe of discourse and D_ω is a given set of possible worlds;[33] $D_{(\alpha\beta)}$ is a set of functions from D_β to D_α. An interpretation maps every constant of type α on an element of D_α; a valuation maps every variable of type α on an element of D_α. If I is an interpretation and V a valuation, then for every expression a of the language we define the intension $||a||_{I,V}$ in the following way. For a constant c, $||c||_{I,V} = I(c)$; for a variable v, $||v||_{I,V} = V(v)$. $||a(b)||_{I,V} = ||a||_{I,V}(||b||_{I,V})$ and $||\lambda x.a||_{I,V}$ is the function which maps every element i of D_β on $||a||_{I,V'}$, where V' is just like V with the single possible exception that it maps x on i.

We can see that Ty2 is merely a minor variation on Ty1. However, we saw that Ty1 was purely extensional; and now it seems that intensional logic boils down to its simple variation — a variation which merely adds one more type and does not affect its basic extensionality in any substantial way. Should we take this as a *reductio ad absurdum* of the possibility of a ('truly') intensional logic? Not really; but we should take it as an indication that a deeper insight into the concept of *intensionality* is needed.

At first, we must distinguish dealing with an abstract, mathematical structure from dealing with something via the structure.[34] It is true that Ty2, in itself, is in no clear way any more intensional than Ty1. However, note that before we can consider a formal system as a *language* and especially as a language underlying a *logic*, we have to single out the category of sentences, and more generally the categories which are to regiment the pre-theoretic categories of expressions of our ordinary language. And here is where an important difference emerges: whereas for Church sentences were expressions of the category o, for an intensional logician they are rather expressions of the category $(o\omega)$. And this difference spreads to the other categories.

Hence the step from an extensional language to an intensional one does not consist in changing the structure of Ty2, but rather in the way of employing it for logical purposes, in the way of matching it with the natural language and pre-formal reasoning.

8 QUINE'S EXTENSIONALIST PROGRAM

Rampant philosophical objections to intensions and intensional logic were raised by W. V. Quine (1960). The basic objection was that intensions were not sufficiently clearcut to be included into the subject matter of a science so rigorous as logic. According to Quine, they do not have real boundaries: there is no clear telling where one ends and another begins. Take propositions, the intensions of sentences: is the proposition expressed by "Berlin is east from Paris" the same as that expressed by "Paris is west of Berlin"? Who is to decide? Similarly, is the intension of the predicate "to be bigger than John" the same as "not to be smaller than John"?

[33] Tichý takes the language of his logic to be fully interpreted, so he does not count with varying universes. (He sees himself as the continuator of the Frege–Russell tradition as discussed by Goldfarb, 1979).

[34] See Peregrin (2000b).

Quine concluded that intensions are simply illusory: there cannot be an entity, he insists, which would not have a clear-cut boundary. (His famous slogan was "No entity without identity.") Hence propositions, properties and intensions in general are non-entities. In comparison, individuals, truth values and sets of individuals, which underlie extensional logic, are objects *par excellence*. At least, that is what Quine claims.

However, did not Carnap and Kripke help us to a *rigorous* concept of intension, which *is* as clearcut as anything can be clearcut at all? True, meanings of expressions of natural languages are fuzzy, but Kripkean intensions are *functions*, and functions are in general well-defined if anything is. It may be an obscure matter whether two sentences of natural language denote the same intension; but intensions themselves are not obscure! Hence are Quine's scruples simply preposterous?

Not really. In fact, the fuzziness Quine diagnosed did find its way even into our notion of intension, hidden within the Trojan horse of the concept of possible world. Is there a possible world in which Berlin is east from Paris, while Paris is *not* west of Berlin? If we imagine possible worlds in some intuitive (in the Kantian sense) way, then it seems not, but could we not have possible worlds with some bizarre non-Euclidean geometries in which something like this would be possible? (And should we insist on *intuitivity* of possible worlds at all?) And again, who is to decide?

What if we construe possible worlds as maximal consistent classes of sentences? Are such worlds not automatically excluded at least in this case? Are not "Berlin is east from Paris" and "Paris is not west of Berlin" incompatible? Well, if all expressions already have exactly specified meanings, then it is determined what is incompatible with what and hence what are the maximal consistent classes of sentences. However, in such a case it is unclear what could be the reason for taking pains to build a possible-worlds semantics: this is an achievement for a language whose semantics is in need of explication, i.e. for a formal language determined merely in terms of axioms and inference rules or for a natural language existing only via the practices of certain communities.

Besides this, even if we denied that the Carnapo-Kripkean intensions are fuzzy, a problem would persist. What we are interested in are *the very intensions of our expressions*; and to promote the goal of their explication it is of little help to have some crisp intensions, about which we are nevertheless unable to say by which real expressions they are denoted. (Lewis, 1975, pointed out that it is one thing to study abstract languages, and it is another thing to pin down the abstract language which can be identified with an empirical language. Our point here is that if what we are interested in is meaning within the empirical languages, then the former enterprise in itself is not of much use.)

Hence Quine seems to have a good point after all: intensions *are* essentially fuzzy. However, be they as fuzzy as they may, the two points stated above which render extensional semantics problematic persist: it is incapable of providing (i) an adequate explication of meanings and (ii) an adequate analysis of non-extensional contexts. What is Quine's response to these obstacles?

As for (i), Quine simply rejects the concept of meaning. This is not to say that he rejects that our expressions are *meaningful*, he only insists that such meaningfulness is a more-or-less matter and as such is not reasonably graspable as a possession, by the expression, of a definite object. "I would not seek," claims Quine (1992, 56), "a scientific rehabilitation of something like the old notion of separate and distinct meanings; the notion is better seen as a stumbling block cleared away."

An objection which can be raised against this view is that the fact that something is fuzzy does not automatically imply that it could not be useful to explicate (or model) it in a non-fuzzy way. After all, everything we encounter within the empirical world is (more or less) fuzzy, and we often profit from making idealized models.[35] The whole point is to be aware of the level of idealization employed and not to employ the model where a higher resolution is required.

As for (ii), Quine claims that some of the alleged non-extensional contexts may be disregarded for their role is in fact marginal; whereas others can be dealt with in some indirect extensional ways. Thus, the contexts studied by ordinary modal logic are, according to him, of the first kind: the contexts *possibly* ... and *necessarily* ..., as understood by modal logicians, are not really important for natural language.[36]

But there are less marginal contexts, like those of propositional attitude reports (without which, for example, no psychological theory would be imaginable). How does Quine want to cope with them? What he proposes is to understand such reports as expressing a relation between an individual and a *sentence*. Thus "John believes that all featherless bipeds are human" will get analyzed as a relation between John and the *sentence* "all featherless bipeds are human". (What if John does not speak English? Would it still make sense to say that his believing that all featherless bipeds are human is a matter of his relationship with an *English* sentence? In one sense yes: he would be related to the sentence by the relation which relates a person to a sentence iff the latter expresses the belief of the former.)

It is now up to the reader to weigh the fuzziness of intensions against the cumbersomeness of the Quinean analysis of intensional contexts.

9 BEYOND INTENSIONS

Anyway, intensional logic has proved itself especially fruitful for the purposes of explicating meanings in natural language[37] (simultaneously with the systems of Montague and Tichý there also appeared others, like that of Creswell, 1973). However, already from the outset it became clear that even intensions are not entirely sufficient for this purpose. This was particularly apparent in the case of propositional attitude reports (mentioned already above). Consider

(10) John believes that one plus one equals two

Viewed from the perspective of intensional semantics, this statement claims that there is a relation of *believing* between a person John and a proposition that one plus one equals two. And the proposition can be nothing else than an intension, namely the intension of "one plus one equals two", hence the class of all possible worlds. So (10) claims that John

[35] See Peregrin (*ibid.*).
[36] See footnote 20.
[37] See Partee & Hendriks (1997).

is in the relation of believing to the class of all possible worlds. But exactly the same is claimed by any sentence which results from (10) by replacing the object clause by any other mathematical truth, i.e.

(11) John believes that every consistent first-order theory has a denumerable model.

This would entail that (10) and (11) are synonymous and especially that they cannot differ in truth value. But while it is unlikely that someone does not believe that one plus one equals two, it is surely possible that he has no idea about properties of first-order theories.[38]

There are several responses to this observation. One of them is not to tie possible worlds to *empirical* possibility. Thus Hintikka (1978) proposed what he called 'impossible possible worlds': such a world is not possible in the sense of being realizable, but is possible in the sense that somebody might simultaneously believe everything which holds in it.

Another, more popular response was to supplement intensional semantics by some superstructure which would allow us to explicate meanings as something more 'fine-grained' than intensions (then we can speak, together with Cresswell (1975), about *hyper-intensional* semantics and perhaps logic). Thus David Lewis (1972), following a hint of Carnap (1947), proposed seeing meanings of compounds as a kind of structures based on the syntactic structures of the corresponding expressions, but involving intensions of their components. These ideas were then elaborated especially by Cresswell (1985). Tichý (1986), in a similar vein, proposed seeing meanings of compounds as 'constructions' of their intensions from the intensions of their parts: hence to see the meaning of *one plus one* as the construction of the number two out of the operation of addition and two instances of the number one; and to see the meaning of *one plus one equals two* as the construction of the truth value **T** out of this construction, the relation of equality and another (trivial) construction of the number two. And propositional attitudes are then construed as relations to constructions.

This is connected to another problem concerning the Carnapian approach to intensions. Having followed Carnap's explication, whereby intensions became in fact extensions relativized to the state of the world, we have arrived at an extension–intension distinction which makes a nontrivial sense only in the case of empirical terms. Where there is no dependence on the state of the world, there the distinction between extension and intension is trivial (the intension becomes a *constant* function mapping every possible world on the same extension). However, it seems that the intuition underlying the Carnapian distinction applies also to mathematical discourse. Hence, should not the distinction between intension and extension apply also to non-empirical vocabulary?

The point behind the 'intuitive' extension and intension (which Carnap set out to explicate) seems to be that whereas on the extensional level we can only say what (actually) 'falls under' a word, on the level of intension we can say *why* it is that it falls under it. The Carnapian explication exploits the idea that the *knowing why* can be explicated as *knowing the way extension depends on the state of the world*. We can know what the sum of

[38]The problems posed by the propositional attitude reports were pointed out already by Carnap (1947); in the post-Montagovian era the discussion was revived by Lewis (1972), Partee (1982), Cresswell (1985) and others.

two numbers is without knowing why; and the knowing why, understanding what addition is, can possibly be equated with the ability to give the sum of *arbitrary* numbers. Similarly, knowing why some things fall under a term, knowing the corresponding concept, is equated with the ability to give the extension of the term w.r.t. arbitrary circumstances.

From this viewpoint, it would seem that minimally within the realm of mathematics, it is something of the kind of Tichyan constructions, rather than Carnapian intensions that should play the role of meanings in the intuitive sense.[39]

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BIBLIOGRAPHY

[Ajdukiewicz, 1935] Ajdukiewicz, K. (1935): 'Die syntaktische Konexität', *Studia Philosophica* 1, 1-27.

[Andrews, 1986] Andrews, P. B. (1986): *Introduction to Mathematical Logic and Type Theory*, Academic Press, San Diego; second edition Kluwer, Dordrecht, 2002.

[Bar-Hillel, 1950] Bar-Hillel, Y. (1950): 'On Syntactical Categories', *Journal of Symbolic Logic* 15, 1-16.

[Bar-Hillel, 1953] Bar-Hillel, Y. (1953): 'A Quasi-arithmetical Notation for Syntactic Description', *Language* 29, 47-58.

[van Benthem and ter Meulen, 1997] Benthem, J. van and A. ter Meulen (1997): *Handbook of Logic and Language*, Elsevier / MIT Press, Oxford / Cambridge (Mass.).

[Blackburn *et al.*, 2000] Blackburn, P., M. de Rijke and Y. Venema (2000): *Modal Logic*, Cambridge University Press, Cambridge.

[Boolos, 1979] Boolos, G. (1979): *The Unprovability of Consistency*, Cambridge University Press, Cambridge.

[Carnap, 1947] Carnap, R. (1947): *Meaning and Necessity*, University of Chicago Press, Chicago.

[Casadio, 1988] Casadio, C. (1988): 'Semantic Categories and the Development of Categorial Grammars', in *Categorial Grammars and Natural Language Structures* (ed. T. Oeherle et al.), Reidel, Dordrecht, 95-123.

[Chellas, 1980] Chellas, B. F. (1980): *Modal Logic: An Introduction*, Cambridge University Press, Cambridge.

[Church, 1940] Church, A. (1940): 'A Formulation of the Simple Theory of Types', *Journal of Symbolic Logic* 5, 56-68.

[Church, 1956] Church, A. (1956): *Introduction to Mathematical Logic*, Princeton University Press, Princeton.

[Copeland, 2002] Copeland, B. J. (2002): 'The Genesis of Possible Worlds Semantics', *Journal of Philosophical logic* 31, 99-137.

[Cresswell, 1975] Cresswell, M. J. (1975): 'Hyperintensional Logic', *Studia Logica* 34, 25-38.

[Cresswell, 1985] Cresswell, M.J. (1985): *Structured Meanings: The Semantics of Propositional Attitudes*, MIT Press, Cambridge (Mass.).

[Divers, 2002] Divers, J. (2002): *Possible Worlds*, Routledge, London.

[Dummett, 1973] Dummett, M. (1973): *Frege, the Philosophy of Language*, Duckworth, London.

[Dummett, 1981] Dummett, M. (1981): *The Interpretation of Frege's Philosophy*, Duckworth, London.

[Etchemendy, 1990] Etchemendy, J. (1990): *The Concept of Logical Consequence*, Harvard University Press, Cambridge (Mass.).

[Frege, 1891] Frege, G. (1891): *Function und Begriff (Ein Vortrag, gehlaten inder Sitzung vom 9.1.1891 der Jenaischen Gesellschaft für Medizin und Naturwissenschaft)*, Jena.

[Frege, 1892a] Frege, G. (1892a): 'Über Begriff und Gegenstand', *Vierteljahrschrift für wissentschaftliche Philosophie* 16, 192-205.

[Frege, 1892b] Frege, G. (1892b): 'Über Sinn und Bedeutung', *Zeitschrift für Philosophie und philosophische Kritik* 100, 25-50.

[Frege, 1893] Frege, G. (1893; 1903): *Grundgesetze der Arithmetik I-II*, Pohle, Jena.

[39] For an elaboration of the approach of Tichý in this direction see Materna (1998).

[Frege, 1983] Frege, G. (1983): *Nachgelassene Schriften* (ed. H. Hermes et al.), Meiner, Hamburg.
[Gallin, 1975] Gallin, D. (1975): *Intensional and Higher-order Modal Logic*, North-Holland, Amsterdam.
[Gamut, 1991] Gamut, L. T. F. (1991): *Logic, Language and Meaning*, vol. 2, University of Chicago Press, Chicago.
[Goldblatt, 1975] Goldblatt, R. (1975): 'First-order Definability in Modal Logic', *Journal of Symbolic Logic* 40, 35-40.
[Goldfarb, 1979] Goldfarb, W.D. (1979): 'Logic in the Twenties: the Nature of the Quantifier', *Journal of Symbolic Logic* 44, 351-368.
[Henkin, 1950] Henkin, L. (1950): 'Completeness in the Theory of Types', *Journal of Symbolic Logic* 15, 81-91.
[Hintikka, 1969] Hintikka, J. (1969): *Models for Modalities*, Reidel, Dordrecht.
[Hintikka, 1978] Hintikka, J. (1978): 'Impossible Possible Worlds Vindicated', in *Game-Theoretical Semantics* (ed. E. Saarinen), Reidel, Dordrecht, 367-379.
[Hughes and Cresswell, 1968] Hughes, G. E. and M. J. Cresswell. (1968): *An Introduction to Modal Logic*, Methuen, London; revised edition *A New Introduction to Modal Logic*, Routledge, London, 1996.
[Janssen, 1997] Janssen, T.M.V. (1997): 'Compositionality', in van Benthem and ter Meulen (1997).
[Janssen, 2001] Janssen, T.M.V. (2001): 'Frege, Contextuality and Compositionality', *Journal of Logic, Language and Information* 10, 115-136.
[Kripke, 1963] Kripke, S. (1963): 'Semantical Analysis of Modal Logic I (Normal Modal Propositional Calculi)', *Zeitschift für mathematische Logik und Grundlagen der Mathematik* 9, 67-96.
[Kripke, 1965] Kripke, S. (1965): 'Semantical Analysis of Modal Logic II (Non-Normal Modal Propositional Calculi)', in *The Theory of Models* (ed. J. W. Addison et al.), North-Holland, Amsterdam, 206-220.
[Lewis, 1932] Lewis, C. I. and C. H. Langford (1932): *Symbolic Logic*, Century, New York.
[Lewis, 1968] Lewis, D. K. (1968): 'Counterpart Theory and Quantified Modal Logic', *Journal of Philosophy* 65, 113-126.
[Lewis, 1972] Lewis, D. K. (1972): 'General Semantics', in *Semantics of Natural Language* (ed. D. Davidson & G. Harman), Reidel, Dordrecht.
[Lewis, 1975] Lewis, D. K. (1975): 'Languages and Language', in *Minnesota Studies in the Philosophy of Science* VII (ed. K. Gunderstone), University of Minnesota Press, Minneapolis.
[Lewis, 1986] Lewis, D. K. (1986): *On the Plurality of Worlds*, Blackwell, Oxford.
[Loux, 1979] Loux, M. J., ed. (1979): *The Possible and the Actual: Readings in the Metaphysics of Modality*, Cornell University Press, Ithaca.
[Materna, 1998] Materna, P. (1998): *Concepts and Objects (Acta Philosophica Fennica 63)*, Societas Philosophica Fennica, Helsinky.
[Mendelson, 1964] Mendelson, E. (1964): *Introduction to Mathematical Logic*, Wadsworth & Brooks, Pacific Grove.
[Montague, 1970a] Montague, R. (1970a): 'English as a Formal Language', in Visenti, B. *et. al.* (eds): *Linguaggi nella Società et nella Tecnica*, Milan, 1970, 189-224; reprinted in Montague (1974).
[Montague, 1970b] Montague, R. (1970b): 'Universal Grammar', *Theoria* 36, 373-398; reprinted in Montague (1974).
[Montague, 1973] Montague, R. (1973): 'The Proper Treatment of Quantification in Ordinary English', in *Approaches to Natural Language*, (ed. J. Hintikka et. al.), Reidel, Dordrecht, 221-242; reprinted in Montague (1974).
[Montague, 1974] Montague, R. (1974): *Formal Philosophy: Selected Papers of R. Montague* (ed. R. Thomason), Yale University Press, New Haven.
[Neale, 2001] Neale, S. (2001): *Facing Facts*, Clarendon Press, Oxford.
[Partee and Hendricks, 1997] Partee, B. and H. L. W. Hendricks (1997): 'Montague Grammar', in van Benthem a ter Meulen (1997), 5-92.
[Partee, 1982] Partee, B. (1982): 'Belief Sentences and Limits of Semantics', in *Processes, Beliefs and Questions* (eds. S. Peters and E. Saarinen), Reidel, Dordrecht.
[Partee, 1976] Partee, B., ed. (1976): *Montague Grammar*, Academic Press.
[Pelletier, 2001] Pelletier, F.J. (2001): 'Did Frege Believe Frege's Principle?', *Journal of Logic, Language and information* 10, 87-114.
[Peregrin, 2000] Peregrin, J. (2000b): 'The "Natural" and the "Formal"', *Journal of Philosophical Logic* 29, 75-101
[Peregrin, 2005] Peregrin, J. (2005): 'Is Compositionality an Empirical Concept?', in M. Werning, E. Machery & G. Schurz (eds.): *The Compositionality of Concepts and Meanings*, Ontos, Frankfurt, 135-150.
[Peregrin, 2006] Peregrin, J. (2006): 'Semantics as Based on Inference', in *The Age of Alternative Logics* (ed. J. van Benthem et al.), Kluwer, Dordrecht, to appear.

[Quine, 1992] Quine, V.W.O. (1992): *Pursuit of Truth* (*revised edition*), Harvard University Press, Cambridge (Mass.).

[Quine, 1960] Quine, W.V.O. (1960): *Word and Object*, MIT Press, Cambridge (Mass.).

[Shapiro, 1991] Shapiro, S. (1991): *Foundations without Foundationalism*, Clarendon Press, Oxford.

[Tarski, 1936] Tarski, A. (1936): 'Über den Begriff der logischen Folgerung', *Actes du Congrès International de Philosophique Scientifique* 7, 1-11; English translation 'On the Concept of Logical Consequence' in Tarski: *Logic, Semantics, Metamathematics*, Clarendon Press, Oxford, 1956, pp. 409-420.

[Tichý, 1971] Tichý, P. (1971): 'An Approach to Intensional Analysis', *Nôus* 5, 273-297; reprinted in Tichý (2004).

[Tichý, 1975] Tichý, P. (1975): 'What do We Talk about?', *Philosophy of Science* 42, 80-93; reprinted in Tichý (2004).

[Tichý, 1978] Tichý, P. (1978): 'De Dicto and De Re', *Philosophia* 8; reprinted in Tichý (2004).

[Tichý, 1978] Tichý, P. (1978): 'Two Kinds of Intensional Logic', *Epistemologia* 1, 143-164; reprinted in Tichý (2004).

[Tichý, 1986] Tichý, P. (1986): 'Constructions', *Philosophy of Science* 53, 514-534; reprinted in Tichý (2004).

[Tichý, 2004] Tichý, P. (2004): *P. Tichý's Collected Papers in Logic and Philosophy* (ed. V. Svoboda, B. Jespersen & C. Cheyne), Filosofia, Prague.

[Zimmerman, 1989] Zimmermann, T.E. (1989): 'Intensional Logic and Two-sorted Type Theory', *The Journal of Symbolic Logic* 54, 65-77.