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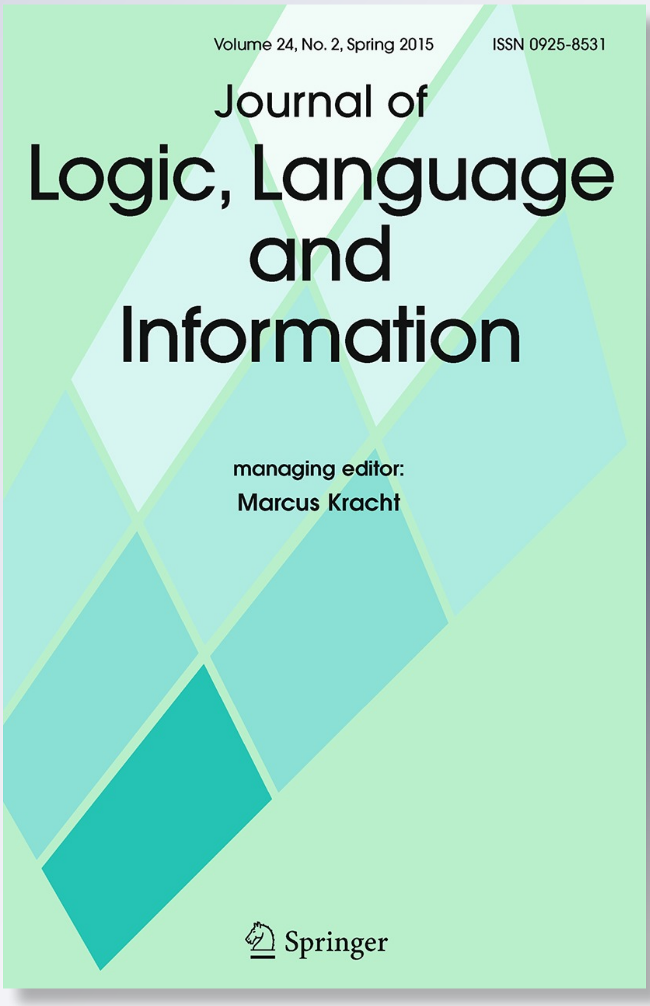

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Abstract What is a minimal proof-theoretical foundation of logic? Two different ways to answer this question may appear to offer themselves: reduce the whole of logic either to the relation of inference, or else to the property of incompatibility. The first way would involve defining logical operators in terms of the algebraic properties of the relation of inference—with conjunction $A \wedge B$ as the *infimum* of A and B, negation $\neg A$ as the *minimal incompatible* of A, etc. The second way involves introducing logical operators in terms of the relation of incompatibility, such that X is incompatible with $\{\neg A\}$ iff every Y incompatible with X is incompatible with $\{A\}$; and X is incompatible with $\{A \wedge B\}$ iff X is incompatible with $\{A,B\}$; etc. Whereas the first route leads us naturally to intuitionistic logic, the second leads us to classical logic. The aim of this paper is threefold: to investigate the relationship of the two approaches within a very general framework, to discuss the viability of erecting logic on such austere foundations, and to find out whether choosing one of the ways we are inevitably led to a specific logical system.

Keywords Inference · Incompatibility · Proof theory · Intuitionistic logic

1 Incompatibility and Inference

Which concept or concepts is a logician to take as primitive? Is there a minimal set of "unexplained explainers" that logic is to rest upon? One answer to these questions would invoke the concept of *truth*. Once we have this concept, we can define truth-functions and hence the whole of classical propositional logic. In order to move on to

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predicate logic, we would then need to switch, as Tarski taught us, to the more general concept of *satisfaction*. This would be a semantic or model-theoretic way to the bare bones of logic.

However, in this paper we want to explore the alternative, proof-theoretic path. Such explorations commonly lead to the relation of *inference*. It is clear that the relation of inference can be seen as a preorder on the set of sentences (A $\vdash$ B being interpreted as A $\geq$ B[1]) and if we take sentences A and B such that both A $\vdash$ B and B $\vdash$ A as identical (thereby, in effect, moving from sentences to something like propositions), we have a partial order. And given this, the usual logical compounds can be defined, very naturally, in its terms: the conjunction of A and B as their *supremum*, the disjunction as their *infimum* etc. The logical operators, then, can be seen as functions mapping propositions on this kind of "extrema" (hence the presence of the explicit operators guarantees that the poset of propositions forms a lattice—the operators are capable of "forging" all the suprema and infima). Adding negation which would be able to map a proposition on its complement would turn the lattice into a Boolean algebra—but adding this kind of negation is not so straightforward. It is much more straightforward to introduce intuitionistic negation and hence to have intuitionistic logic.[2]

Another concept to which we may attempt to reduce logic is incompatibility (this concept is not "proof-theoretical" in such a straightforward sense as inference, which amounts to precisely the relation we make explicit by proofs; but nevertheless it clearly belongs to the realm of what Carnap would call "logical syntax"[3] and it thus also stands in opposition to semantics or model theory). This kind of reduction was attempted by Brandom and Aker (2008). They defined what they called an *incompatibility frame*, consisting, in effect, of a set (of sentences) plus a set of its subsets closed to forming supersets. They characterize logical operators in terms of their "incompatibility conditions"—*viz*. conditions under which a set is incompatible with a sentence formed by means of the operator. And the outcome of taking this attitude to logic is classical logic. Does this mark an inherent difference between inference and incompatibility, and hence the logics based upon each, in that the first is somehow predestined to yield intuitionistic logic, while the other yields classical logic?

To compare these approaches, we will consider structures possessing both inference and incompatibility independent of each other. Consider the ordered triple $<S, \perp, \vdash>$, where S is a countable set, $\perp \subseteq \mathrm{Pow}(S)$, and $\vdash \subseteq \mathrm{Pow}(S) \times S$. Which constraints should be placed on the notions of incompatibility ($\perp$) and inference ($\vdash$) on this very general level? (It is clear that not any kind of set of sets of sentences can be

[1] It might seem more natural to interpret inference taking the reverse perspective: i.e. A $\vdash$ B as A $\leq$ B (especially if we tend to see propositions as classes of possible worlds), but we will stick to the perspective adopted by most logicians (including both Brandom and Koslow, whose texts are starting points of our considerations).

[2] Koslow's (1992) approach is closely connected. However, he considers all logical compounds as *minima* of certain functions w.r.t. this ordering.

[3] As Carnap (1934, p. 2) puts it: "We shall see that the logical characteristics of sentences (for instance, whether a sentence is analytic, synthetic, or contradictory; whether it is an existential sentence or not; and so on) and the logical relations between them (for instance, whether two sentences contradict one another or are compatible with one another; whether one is logically deducible from the other or not; and so on) are solely dependent upon the syntactical structure of the sentences."

reasonably seen as instantiating incompatibility, and that not every relation between sets of sentences and sentences can be reasonably called a relation of inference.)

Before we introduce the basic constraints, a word about notation. The variables X, Y, Z will range over subsets of S, while the variables A, B, C will range over elements of S. $\perp$X will be shorthand for $X \in \perp$, $X \vdash A$ for $< X, A > \in \vdash$. X,Y will be shorthand for X$\cup$Y and A will be shorthand for {A}. Hence, e.g. $\perp$X,A expands to $X \cup \{A\} \in \perp$. Now we can list the constraints we will take for basic.

($\perp$) for every X,Y: if $\perp$X and $X \subseteq Y$, then $\perp$Y
($\vdash$1) for every X,A: X,A $\vdash$ A
($\vdash$2) for every X,Y,A,B: if X,A $\vdash$ B and Y $\vdash$ A, then X,Y $\vdash$ B

Let us adopt a further notational convention. Symbols that appear as "free" in the conditions of the above kind will be understood as universally quantified. Given this convention, we can shorten the above conditions to

($\perp$) if $\perp$X and $X \subseteq Y$, then $\perp$Y
($\vdash$1) A,X $\vdash$ A
($\vdash$2) if X,A $\vdash$ B and Y $\vdash$ A, then X,Y $\vdash$ B

($\perp$) states that an incompatible set of sentences cannot be turned into a compatible one by adding further sentences. This is the single constraint stipulated by Brandom and Aker. ($\vdash$1) states that sentences that belong to a set are inferable from the set. ($\vdash$2) says that the relation of inference is transitive. The constraints ($\vdash$1) and ($\vdash$2) are clearly tantamount to the Gentzenian structural rules.[4]

The ordered triple $< S, \perp, \vdash >$, where S is a set $\perp \subseteq$ Pow(S) and $\vdash \subseteq$ Pow(S) $\times$ S, will be called a (*standard*[5]) *generalized inferential structure* (*gis*) iff it complies with ($\vdash$1), ($\vdash$2) and ($\perp$).

Can we consider any interrelation between inference and incompatibility, or can we even reduce one to the other? As for reducing incompatibility to inference, the only possibility, at this quite general level, is to consider a set of sentences incompatible iff everything can be inferred from it ("ex contradictione quodlibet"). Such a reduction is tantamount to the following stipulation

[4] Gentzen (1934; 1936) introduced structural rules by means of which he characterized those relations of inference that he took to be "standard". In a slightly more contemporary manner, they can be summarized as restrictions on the relation $\vdash$ between finite sequences of sentences and sentences as follows:

A $\vdash$ A	(*reflectivity*)
if X,Y $\vdash$ A, then X,B,Y $\vdash$ A	(*weakening* or *extension*)
if X,A,A,Y $\vdash$ B, then X,A,Y $\vdash$ B	(*contraction*)
if X,A,B,Y $\vdash$ C, then X,B,A,Y $\vdash$ C	(*permutation* or *exchange*)
if X,A,Y $\vdash$ B and Z $\vdash$ A, then X, *Z*, Y $\vdash$ B	(*cut*)

Within our framework, two of the conditions, namely *contraction* and *permutation*, are implicit to our assumption that inference is a relation between *sets* (rather than sequences) of sentences and sentences. *Reflectivity* is obviously a special case of ($\vdash$1), *cut* is embodied in ($\vdash$2) and weakening follows from ($\vdash$1) and ($\vdash$2): if X $\vdash$ A, then as A,B $\vdash$ A by ($\vdash$1), X,B $\vdash$ A follows by ($\vdash$2).

[5] This qualification, which will be omitted in this paper as we will deal only with standard gis's, is included because it would be possible to study structures in which some of the conditions ($\vdash$1), ($\vdash$2) and ($\perp$)—and hence some of Gentzen's structural rules—would be relaxed, thus entering the realm corresponding to that of substructural logics.

($\bot\vdash$1) if $\bot$X, then X $\vdash$ A
($\bot\vdash$2) if X $\vdash$ A for every A, then $\bot$X

As for reducing, the other way around, inference to incompatibility, again, on this general level the only viable possibility appears to be to take A to be inferable from X iff whatever is incompatible with A is incompatible with X. This would amount to

($\vdash\bot$1) if X $\vdash$ A and $\bot$Y,A, then $\bot$Y,X
($\vdash\bot$2) if $\bot$Y,A implies $\bot$Y,X for every Y, then X $\vdash$ A[6]

Let us call a set X of sentences *quasiincompatible* iff X $\vdash$ A for every A; we will write $\triangle$X. And let us call A *quasiinferable* from X iff $\bot$Y,A implies $\bot$Y,X for every Y; we will denote this as X $\triangleright$ A. Then we can say that ($\bot\vdash$1) and ($\bot\vdash$2) state the equivalence of incompatibility and quasiincompatibility (the first of them states that quasiincompatibility is a necessary condition of incompatibility, the second one states that it is a sufficient condition); while ($\vdash\bot$1) and ($\vdash\bot$2) state the equivalence of inferability and quasiinferability (again, the first of them states that quasiinferability is a necessary condition of inferability, the second one states that it is a sufficient condition).

We do not claim that such interdefinability of $\bot$ and $\vdash$ is inevitable, or that it is plausible. Many logicians would surely protest that to identify incompatibility with entailing everything is *not* plausible; and similar objections would be probably raised against the converse reduction. But I want to point out that *if* we want to have only *one* basic (irreducible) concept, *then* on this general level we cannot but accept at least one of these reductions; and also I want to explore the consequence of adopting them. So let us call a gis *normal* iff it complies with ($\vdash\bot$1), ($\vdash\bot$2), ($\bot\vdash$1), and ($\bot\vdash$2).

Hence a normal gis complies with ($\vdash$1), ($\vdash$2), ($\bot$), ($\vdash\bot$1), ($\vdash\bot$2), ($\bot\vdash$1), and ($\bot\vdash$2). Now some of these constraints turn out to be superfluous—we present the most basic facts without proofs[7]:

Theorem 1 *$<S, \bot, \vdash>$ is a normal gis iff it complies with* ($\vdash$1), ($\vdash$2), ($\bot$), ($\vdash\bot$2)*, and* ($\bot\vdash$2).

Theorem 2 *In a normal gis $\triangle$ coincides with $\bot$ and $\triangleright$ coincides with $\vdash$.*

The normalness of a gis is a matter of constraints on the interplay of $\vdash$ and $\bot$; but does it, in some way, constrain also $\vdash$ or $\bot$ alone? That is, does it imply some restriction concerning either $\vdash$ or $\bot$ alone, not implied by either ($\vdash$1) and ($\vdash$2) or ($\bot$), respectively?

It is clear that if we reduce $\bot$ to $\vdash$, by means of ($\bot\vdash$1) and ($\bot\vdash$2), and then go on and reduce $\vdash$ back to $\bot$, by means of ($\vdash\bot$1) and ($\vdash\bot$2), we get a constraint concerning $\bot$ alone, namely

($\bot'$)$\bot$X iff X $\triangleright$ A for every A,

which says that X is incompatible iff everything is quasiinferable from it. As it turns out, ($\bot'$) does not follow from ($\bot$), but it can be reduced to

[6] Brandom and Aker call this condition—more precisely an equivalent one—*defeasibility*.

[7] They can be found in Peregrin (2011).

($\perp''$)$\perp$S

Theorem 3 *A gis* $<S, \perp, \vdash>$ *complies with* ($\perp'$) *iff it complies with* ($\perp''$).

Proof Let first $<S, \perp, \vdash>$ comply with ($\perp'$). Assume that $\perp = \varnothing$ (i.e. that no subset of S is incompatible). Then for no $A \in S$ and no $Y \subseteq S$ it is the case that $\perp$Y,A, and hence for every $A \in S$ and every $X,Y \subseteq S$ it is (trivially) the case that $\perp$Y,A implies $\perp$Y,X. But then, according to ($\perp'$), $\perp$X for every $X \subseteq S$. As this contradicts the assumption that $\perp = \varnothing$, it must be the case that $\perp$X for some $X \subseteq S$. But then, in force of ($\perp$), $\perp$S.

Now let $<S, \perp, \vdash>$ comply with ($\perp''$). First assume that $\perp$X. Then, in force of ($\perp$), $\perp$Y,X for every $Y \subseteq S$, and hence $\perp$Y,A implies $\perp$Y,X for every $A \in S$ and every $Y \subseteq S$. Next assume that $X \rhd A$ for every A, i.e. that $\perp$Y,A entails $\perp$Y,X for every Y. It follows that for every Y and every A, if not $\perp$Y,X, then not $\perp$Y,A; and hence especially (considering only supersets of X, i.e. instantiating Y as Z,X) that for every Z and every A, if not $\perp$Z,X, then not $\perp$Z,X,A. Let $S = \{A_1, A_2, A_3, \ldots\}$ (remember that S is countable). Assume that not $\perp$X, we will prove, by induction, that not $\perp$S. As not $\perp$X, not $\perp$X,A_1. Moreover, if not $\perp$X,$A_1, \ldots, A_n$, then not $\perp$X,$A_1, \ldots, A_n, A_{n+1}$. Hence not $\perp$X,$A_1, A_2, A_3, \ldots$, and hence that not $\perp$S. This is a contradiction and hence it cannot be the case that not $\perp$X. □

Thus, in a normal structure, not all sets of sentences are compatible. It is helpful to have a term for a structure in which not all sets of sentences are *in*compatible; hence let us call such a structure *consistent*.

In case of $\vdash$, we can reduce $\vdash$ to $\perp$, by means of ($\vdash\perp$1) and ($\vdash\perp$2), and then go on and reduce $\perp$ back to $\vdash$, by means of ($\perp\vdash$1) and ($\perp\vdash$2), we get

($\vdash$3) $X \vdash A$ iff if $\triangle$Y,A implies $\triangle$Y,X for every Y.

In our terminology, it says that A is inferable from X iff everything that is quasiincompatible with A is quasiincompatible with X. Again, this constraint does not follow from ($\vdash$1) and ($\vdash$2) alone.

Theorem 4 *There is a gis in which* ($\vdash$3) *fails.*

Proof Let $<S, \perp, \vdash>$ be the gis such that $S = \{\alpha, \beta, \gamma\}$, $\perp = \varnothing$, and $\vdash$ be the closure of $\alpha \vdash \beta$ and $\beta \vdash \gamma$ under ($\vdash$1) and ($\vdash$2), i.e. all the valid instances of $\vdash$ are

$\alpha, \beta, \gamma \vdash \alpha$	$\alpha, \beta, \gamma \vdash \beta$	$\alpha, \beta, \gamma \vdash \gamma$
$\alpha, \beta \vdash \alpha$	$\alpha, \beta \vdash \beta$	$\alpha, \beta \vdash \gamma$
$\beta, \gamma \vdash \beta$	$\beta, \gamma \vdash \gamma$	
$\alpha, \gamma \vdash \alpha$	$\alpha, \gamma \vdash \beta$	$\alpha, \gamma \vdash \gamma$
$\alpha \vdash \alpha$	$\alpha \vdash \beta$	$\alpha \vdash \gamma$
$\beta \vdash \beta$	$\beta \vdash \gamma$	$\gamma \vdash \gamma$

Then it is easy to check that the gis does not comply with ($\vdash$3). First, observe that every subset of S which is quasiincompatible with β (i.e. $\{\alpha\}$, $\{\alpha, \beta\}$, $\{\alpha, \gamma\}$, $\{\alpha, \beta, \gamma\}$) is quasiincompatible with γ. Then, if $<S, \perp, \vdash>$ were to comply with ($\vdash$3), it would follow that $\gamma \vdash \beta$, which is not the case. □

Notice that while if it is inference that we take for the single primitive notion, we are free to set it up so that (⊢3) does, or does not hold; if it is incompatibility and inference is derived from it, this option is not available—(⊢3) holds just by way of deriving inference from incompatibility.

2 Negation, Conjunction and Implication

Let us now turn our attention to logical operators.

First consider a definition inspired by that put forward by Koslow (1992, p. 91)[8]:

(¬K1)△A,¬A
(¬K2) if △A,X, then X ⊢ ¬A

Koslow shows that his definition yields negation which is intuitionistic, and not necessarily classical, as it does not necessarily yield

(¬¬) ¬¬A ⊢ A.

Koslow's counterexample is the structure we used in the proof of Theorem 4, which yields us $\neg\alpha = \gamma$, $\neg\beta = \alpha$, and $\neg\gamma = \alpha$, and it is easily seen that it works for our definition as well.

This is, of course, not surprising—the intimate connection between single-conclusion inference and intuitionistic logic is clear.[9] What do we need to add if we were to want the classical negation? Of course we can add directly (¬¬). But it turns out that there are other, more subtle, modifications of the Koslowian definition to the same effect. One of them is to replace (¬K2) by

(¬K3) if △¬A,X, then X ⊢ A

This constraint stipulates that the negation of A is a sentence whose *minimal quasiincompatible* is A.

Theorem 5 *Let* $<S, \perp, \vdash>$ *be a gis for which* (¬*K*1) *and* (¬*K*3) *hold. Then both* (¬¬) *and* (¬*K*2) *hold.*

Proof According to (¬K3), it is the case that if △¬A,¬¬A, then ¬¬A ⊢ A; and △¬A,¬¬A follows from (¬K1). Hence (¬¬) holds. It follows that for every X, if △X,A, then △X,¬¬A; and hence, according to (¬K3), that if △X,A, then X ⊢ ¬A. Hence (¬K2) holds. □

Let us note that (¬K1) + (¬K2), resp. (¬K1) + (¬K3) are equivalent to

(¬K)△A,X iff X ⊢ ¬A, resp.
(¬K*) △¬A,X iff X ⊢ A

[8] Koslow uses, in effect, "if △A,B, then B ⊢ ¬A" instead of our (¬K2). Our definition is more general, but in the presence of conjunction or implication, the difference is significant only in cases where the incompatibility or inference dealt with is not compact, which is not a case we will be interested in here.

[9] See Peregrin (2008) for a discussion of this.

Lemma 6 *In a gis,* $(\neg K1) + (\neg K2)$ *hold iff* $(\neg K)$ *holds; and* $(\neg K1) + (\neg K3)$ *hold iff* $(\neg K^*)$ *holds.*

Proof Let (¬K1) + (¬K2) hold. As the direct implication of (¬K) is identical with (¬K2), we must prove only the indirect one. Hence assume X ⊢ ¬A. Then in view of (¬K1) and (⊢2), △A,X. Let, conversely, (¬K) hold. As ¬A ⊢ ¬A, according to (⊢1), it is the case that △¬A,A and (¬K1) holds. (¬K2) holds trivially.

Let now (¬K1) + (¬K3) hold. The direct implication of (¬K*) is again identical with (¬K3); hence assume X ⊢ A. Then in view of (¬K1) and (⊢2), △¬A,X. Let, conversely, (¬K*) hold. As A ⊢ A, according to (⊢1), it is the case that △¬A,A and (¬K1) holds. (¬K3) holds trivially. □

Is there a reason to prefer (¬K2) to (¬K3), or does getting intuitionistic or classical negation result just from our wholly arbitrary choice? A reason is that while (¬K2) fits with the Gentzenian conception of defining logical operators, (¬K3) does not. (The Gentzenian orthodoxy is that an operator is determined by introduction rules plus elimination rules, where the latter ones are kind of secondary, because there is a sense in which they are "contained in" the former.[10])

Let us now turn our attention to *normal* structures. The fact is that within a normal structure, (¬K2) entails (¬K3); and hence already the intuitionist definition yields us classical negation.

Theorem 7 *Let* $< S, \perp, \vdash >$ *be a normal gis for which* $(\neg K1)$ *and* $(\neg K2)$ *hold. Then both* $(\neg\neg)$ *and* $(\neg K3)$ *hold.*

Proof Let (¬K1) and (¬K2), and hence (¬K), hold. Let △A,X. Then, according to (¬K2), X ⊢ ¬A, and hence, according to (⊢⊥1), for every Y, if △¬A,Y then △X,Y. In particular it holds that if △¬A,¬¬A then △X,¬¬A. But △¬A,¬¬A holds according to (¬K1), and hence △X,¬¬A, Thus, △X,A entails △X,¬¬A for every X, and hence, according to (⊢⊥1), ¬¬A ⊢ A.

Now assume △¬A,X. Then, according to (¬K2), X ⊢ ¬¬A, and hence, according to (¬¬) and (⊢2), X ⊢ ¬A. □

This indicates that it is already (⊢3), which holds in every normal structure, that provides for the transformation of (¬K1) plus (¬K2) into the definition of classical, rather than intuitionistic negation.

Now consider the definition of negation in terms of incompatibility put forward by Brandom and Aker:

(BA) ⊥¬A,X iff X ⊳ A.

We can see that this is a counterpart of (¬K*) (and indeed in a normal structure, it is *nothing else* than (¬K*)). From this viewpoint it is clear why Brandom and Aker reach classical negation.

Now a natural idea how to define a negation that would not be classical, but rather intuitionistic, in terms of incompatibility would be to copy our (¬K):

10 See Peregrin (2008). Koslow makes this containment more explicit by replacing the elimination rules by his "extremality conditions".

(BA*) $\perp$A,X iff X $\triangleright$ ¬A

Does this really yield us merely intuitionist negation? A way to find out would be to consider the validity of (¬¬). But what is "⊢" in ¬¬A ⊢ A supposed to represent here? The natural answer, namely that it is $\triangleright$, entails that the gis we are considering is normal, hence that (¬¬) holds, and hence that ¬ is classical. This corresponds to our above observation that if our basic notion is incompatibility and we derive inference from it, then we have (⊢3); and (⊢3) turns negation defined intuitionistically into the classical one.

Could we answer the question whether (BA*) yields us classical negation completely disregarding the concept of inference (thus avoiding bringing in (⊢3))? One way would be to check whether from the viewpoint of incompatibility, ¬¬A is equivalent to A, i.e. whether it is the case that $\perp$A,X iff $\perp$¬¬A,X for every X. And the answer is, of course positive: this follows from the fact that, as is easily seen, not only ¬¬A $\triangleright$ A, but also A $\triangleright$ ¬¬A.

This indicates that while if our sole basic notion is inference, then we can choose whether we want classical or intuitionistic negation (where it is the latter that comes more naturally), if the basic notion is incompatibility, we have little choice—our negation will be classical.

Now let us consider conjunction. The most straightforward inferential way of introducing it is as the infimum:

(∧K1) A∧B ⊢ A
(∧K2) A∧B ⊢ B
(∧K3) if X ⊢ A and X ⊢ B, then X ⊢ A∧B

It is easy to see that (∧K3) is, in the context of (∧K1) and (∧K2), equivalent to

(∧K3′) A,B ⊢ A∧B.

Moreover, it can be shown that in a normal gis, (∧K1), (∧K2) and (∧K3) are equivalent to the following definition of conjunction, due to Brandom and Aker[11]:

(∧B1) if $\perp$X,A∧B, then $\perp$X,A,B
(∧B2) if $\perp$X,A,B, then $\perp$X,A∧B

Consider, finally, implication. The standard way of its proof-theoretical definition is (→I) and either (→E), or (→E′):

(→I) if X,A ⊢ B, then X ⊢ A→B
(→E) A, A→B ⊢ B
(→E′) if X ⊢ A→B, then X,A ⊢ B.

This gives us the intuitionist implication. To get the classical implication, we need to either add classical negation, or the axiom known as Peirce's law:

(PL) ((A→B)→A) ⊢ A

Now a straightforward way of defining implication in terms of incompatibility would be

[11] Again, the proofs can be found in Peregrin (2011).

$(\rightarrow)\bot X,A\rightarrow B$ iff $X \vdash A$ and $\bot X,B$

This involves the inferential definition:

Theorem 8 *In a normal structure,* $(\rightarrow)$ *entails both* $(\rightarrow I)$ *and* $(\rightarrow E')$.

Proof Suppose that $(\rightarrow)$ holds; let us first show that $(\rightarrow I)$ holds. Hence let $X,A \vdash B$; we will show that for every Y, $\bot Y,A\rightarrow B$ entails $\bot X,Y$; $X \vdash A\rightarrow B$ then follows according to $(\vdash\bot 2)$. $\bot Y,A\rightarrow B$ gives us, according to $(\rightarrow)$, $Y \vdash A$ and $\bot Y,B$. $X,A \vdash B$ and $\bot Y,B$ yield us, according to $(\vdash\bot 1)$, $\bot X,A,Y$. Given $Y \vdash A$, this further yield us, again according to $(\vdash\bot 1)$, $\bot X,Y,Y$ and hence $\bot X,Y$.

Now let us show that $(\rightarrow E')$ holds, hence assume $X \vdash A\rightarrow B$. According to $(\vdash\bot 1)$, this means that $\bot Y,A\rightarrow B$ entails $\bot Y,X$ for every Y. Using $(\rightarrow)$, we have that $Y \vdash A$ and $\bot Y,B$ entail $\bot Y,X$ for every Y, in particular that $Y,A \vdash A$ and $\bot Y,A,B$ entail $\bot Y,A,X$ for every Y. But as $Y,A \vdash A$ holds for every Y, this means that $\bot Y,A,B$ entails $\bot Y,A,X$ for every Y, and in particular that $\bot Y,B$ entails $\bot Y,A,X$ for every Y. But then, according to $(\vdash\bot 2)$, $A,X \vdash B$. □

What is remarkable, however, is that the converse does *not* hold, namely that implication defined by $(\rightarrow)$ is *classical*. This can be shown by showing that it complies with (PL).

Theorem 9 *In a normal structure,* $(\rightarrow)$ *entails (PL).*

Proof Suppose that $(\rightarrow)$ holds. Then $\bot X,(A\rightarrow B)\rightarrow A$ iff $X \vdash A\rightarrow B$ and $\bot X,A$. But $\bot X,A$, according to $(\bot\vdash 1)$, entails $X,A \vdash B$, which, according to $(\rightarrow E')$ further entails $X \vdash A\rightarrow B$. Hence $\bot X,A$ entails $\bot X,(A\rightarrow B)\rightarrow A$, and thus (PL) follows by $(\vdash\bot 2)$. □

It is, however, possible to define classical implication in terms of inference, by composing implication out of conjunction $((\wedge K1), (\wedge K2), (\wedge K3))$ and classical negation $((\neg K1), (\neg K3))$. However, what we *cannot* do is define intuitionist implication in term of incompatibility. We might think of changing $(\rightarrow)$ to

$(\rightarrow')\bot X,A\rightarrow B$ iff $\bot X,\neg A$ and $\bot X,B$,

where the negation would be the intuitionist one; but we already know that intuitionistic negation is not available within the logic based on incompatibility (in particular that (BA*) gives us *classical* negation).

3 The Semantic Perspective

We have just seen that there is a way of basing logic on inference and a way of basing it on incompatibility; and that in both ways we can reach classical negation and hence classical logic. However, one might object that nevertheless *both* these ways to classical negation are fallacious, for what they reach is a mere *imitation* of classical logic, not the real thing. To reach negation that is *genuinely* classical, so the objection would go, we would need the concept of truth, rather than inference or incompatibility.

Various logicians have pointed out that the axioms of classical logic do not exclude all valuations that are incompatible with the classical truth tables; in particular, they are not capable of excluding all valuations mapping both a formula and its negation on 0 (Carnap 1943, was probably the first to notice this; subsequently there have been occasional discussions in the literature.[12]). This situation concerning classical propositional logic may resemble, at least at first sight, that of second-order predicate logic, where we have two versions of the logic according to whether we take it to be equipped with standard or with Henkin semantics (see Shapiro 1991). However, whereas in the case of second-order logic the different semantic systems yield different sets of tautologies (hence, we have two logics in a strong sense), here the difference is more subtle (making it more appropriate to talk about two variants of a single logic).

Belnap and Massey (1990) call the two kinds of semantics for classical propositional logic *classical* and *inferential*. Proof-theoretically, we can reach the classical semantics by means of the multi-conclusion sequents. Thus, negation classical in the strong sense can be defined by means of the sequents

(L¬) $\neg A, A \vdash$ and
(R¬) $\vdash \neg A, A$.

Allowing for sequents with empty right-hand (though not yet with the right-hand side with more than one formula) amounts to the introduction of incompatibility as a primitive and it can be seen as a step towards the classical semantics (it allows us for stipulating (L¬), but not yet (R¬)); hence it allows us to exclude some of the intuitionistically, but not classically, acceptable valuations, though not all of them.

To enter semantics into our picture, let us assume that our language is furnished with truth-valuations. The set of all valuations of S is the set $V = \{0, 1\}^S$, and a semantics of S renders some of the valuations *admissible* and the others as *inadmissible*; let us denote the set of all admissible valuations as V*. Let us use the sign $\|A\|$ to denote the class of all admissible valuations mapping the formula A on 1. The way of semantics is to single out the admissible valuations explicitly. Thus, we can specify the semantics of classical logical operators by means of truth tables: we stipulate, in effect, that a valuation is admissible iff it maps every conjunction on 1 just in the case it maps both its conjuncts on 1, etc.

From the viewpoint of semantics, inference and incompatibility can be seen as tools for doing the same thing implicitly. Any pattern of inference or incompatibility can be seen as rendering some truth-valuations as inadmissible, hence excluding them from the set of admissible valuations. In particular, the pattern $X \vdash A$ excludes all valuations that map all elements of X on 1 and A on 0, whereas $\perp X$ excludes those which map all elements of X on 1.[13]

A sequent of the form $X \vdash Y$ excludes all valuations that map all elements of X on 1 and at the same time all elements of Y on 0. Such generalized patterns are sufficient to delimit any set of admissible truth-valuations whatsoever.[14] As we already noted, classical negation can be instituted via the sequents (L¬) and (R¬). The first of them

[12] The most extensive discussion I know of can be found in a recent book by Garson (2013).

[13] See Peregrin (2010a). Probably the first to assume this perspective was Scott (1971, 1972).

[14] See Peregrin (2010a).

excludes all valuations that map both ¬A and A on 1; the second excludes all those that map both of them on 0. It follows that the remaining valuations are those that map precisely one of ¬A and A on 1, and hence ¬ follows its classical truth table.

This generalized form of inference is useful also because both inference in the narrow sense and incompatibility can be seen as its special cases. Inference in the narrow sense can be seen as the case of generalized inference in which the consequent is a singleton, while incompatibility can be seen as the case where the consequent is the empty set (indeed $\perp X$ obviously excludes the very same valuations as $X \vdash$).

If we now look at conjunction, we can see that it is delimitable merely by means of inferences in the narrow sense. Indeed, the usual pattern consisting of (∧K1), (∧K2) and (∧K3′) consists merely of inferences in the narrow sense and it excludes all valuations that are not in accordance with the classical truth table. To prepare the ground for our discussion of negation, let us consider the way these inferences pin down the class of admissible valuations of A∧B relative to those of A and B. First, consider

(∧K1) $A \wedge B \vdash A$.

It stipulates that every valuation that maps A∧B on 1 maps also A on 1, hence that $\|A \wedge B\| \subseteq \|A\|$. Similarly

(∧K2) $A \wedge B \vdash B$

stipulates that $\|A \wedge B\| \subseteq \|B\|$. And as

(∧K3′) $A, B \vdash A \wedge B$

stipulates that $\|A\| \cap \|B\| \subseteq \|A \wedge B\|$, it follows that $\|A\| \cap \|B\| = \|A \wedge B\|$ and hence that conjunction acts as the operation of intersection.

Now consider our rules for negation. If we use incompatibility as the primitive notion, we can use the rule

(¬K1*) $\perp A, \neg A$

to exclude all valuations that map both A and ¬A on 1 and hence to stipulate that $\|A\| \cap \|\neg A\| \subseteq \emptyset$; and now we would need something like the rule (R¬), which would stipulate that $\|A\| \cup \|\neg A\| = V^*$. However, as we cannot have directly (R¬) we must make do with approximations.

Note that $\|A\| \cup \|\neg A\| = V^*$ is equivalent to $V^* \subseteq \|A\| \cup \|\neg A\|$ and further both to $V^* \backslash \|\neg A\| \subseteq \|A\|$ and to $V^* \backslash \|A\| \subseteq \|\neg A\|$. The trouble, however, is that we have no guarantee that we have a sentence that would denote V* (i.e. there may not exist a B such that $\|B\| = V^*$), and hence we must approximate it by its greatest subset that is so denoted.

Let V be a class of valuations; let us call the sets $\cap\{\|B\| \mid V \subseteq \|B\|\}$ and $\cup\{\|B\| \mid \|B\| \subseteq V\}$ the ∩-approximation of the set V and the ∪-approximation of V, respectively. We will call a set of valuations ∩-expressible if it equals its ∩-approximation, and we will call it ∪-expressible if it equals its ∪-approximation. A set V will be called expressible iff there is a B such that $\|B\| = V$; it is clear that an expressible set is both ∪-expressible and ∩-expressible.

Given we cannot be sure that either V* or $\|A\| \cup \|\neg A\|$ is expressible, we must approximate $\|A\| \cup \|\neg A\| = V^*$ as

$\cup\{\|C\| \mid \|C\| \subseteq V^*\} \subseteq \cap\{\|B\| \mid \|A\| \cup \|\neg A\| \subseteq \|B\|\}$.

This is, as is easily seen, equivalent to

for every C such that $\|C\| \subseteq V^*$ and for every B such that $\|A\| \cup \|\neg A\| \subseteq \|B\|$ it is the case that $\|C\| \subseteq \|B\|$,

which is further equivalent to

if $\|A\| \subseteq \|B\|$ and $\|\neg A\| \subseteq \|B\|$, then $\|C\| \subseteq \|B\|$ for every B and every C.

If we now allow for $\vdash$ as the second primitive operator, we can express this as

if $\neg A \vdash B$ and $A \vdash B$, then $\vdash B$.

Hence this last condition does the same job as (R$\neg$) iff V^* is $\cup$-expressible and $\|A\| \cup \|\neg A\|$ is $\cap$-expressible.

Alternatively, we can start from $V^*\backslash\|A\| \subseteq \|\neg A\|$, and replacing $V^*\backslash\|A\|$ by its $\cup$-approximation we get

$\cup\{\|B\| \mid \|B\| \subseteq V^*\backslash\|A\|\} \subseteq \|A\|$,

which is equivalent to

for every B such that $\|B\| \subseteq V^*\backslash\|A\|$ it is the case that $\|B\| \subseteq \|\neg A\|$,

and further to

if $\|B\| \cap \|A\| = \varnothing$, then $\|B\| \subseteq \|\neg A\|$.

This is obviously the semantic correlate of

($\neg$K2*) if $\perp$A,B, then B $\vdash \neg$A.

In this case, we can say that this condition does the same job as (R$\neg$) iff $V^*\backslash\|A\|$ is $\cup$-expressible.

Analogously, we can get from $V^*\backslash\|\neg A\| \subseteq \|A\|$ to

if $\|\neg A\| \cap \|C\| = \varnothing$, then $\|C\| \subseteq \|A\|$,

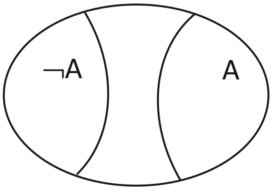
which is the semantic correlate of

($\neg$K3*) if $\perp\neg$A,C, then C $\vdash$ A

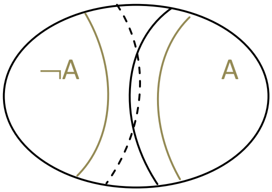
and in this comes to be equivalent to (R$\neg$) iff $V^*\backslash\|\neg A\|$ is $\cup$-expressible.

Now when we do not want to use incompatibility, but only inference, there will be further approximating, for we would have to approximate $\perp$A,B by A,B $\vdash$ C for every C; hence, semantically, $\|A\| \cap \|B\| = \varnothing$ by $\|A\| \cap \|B\| \subseteq \|C\|$ for every C, which would be accurate only iff $\varnothing$ would be $\cap$-expressible. The situation would be similar if we were to use not inference, but only incompatibility.

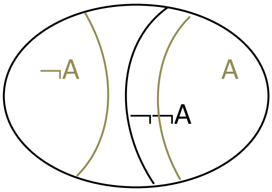
How can we, from the semantic viewpoint, account for the fact that while ($\neg$K1) and ($\neg$K2) do not yield classical logic, ($\neg$K1) and ($\neg$K3) do? ($\neg$K1) stipulates that $\|A\|$ and $\|\neg A\|$ are disjoint (in fact, if we do not have incompatibility but only quasiincompatibility, it stipulates this only on the condition that $\varnothing$ is $\cap$-expressible, for otherwise it can only stipulate that the intersection of $\|A\|$ and $\|\neg A\|$ is contained in the $\cap$-approximation of $\varnothing$). Hence the situation can be depicted as follows:



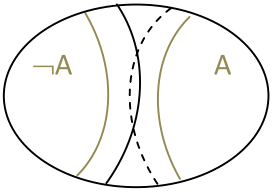
Now if we add (¬K2) we can have a proper superset of ‖A‖ disjoint with ‖¬A‖, while we cannot have a proper superset of ‖¬A‖ disjoint with ‖A‖.



This means that if we have the denotation of the negation of the negation of A, ‖¬¬A‖, it can be a proper superset of ‖A‖ disjoint with ‖¬A‖.



In contrast to this, in the case of (¬K3) we can have a proper superset of ‖¬A‖ disjoint with ‖A‖, while we cannot have a proper superset of ‖A‖ disjoint with ‖¬A‖



It follows that ‖¬¬A‖ cannot be a proper superset of ‖A‖ disjoint with ‖¬A‖, hence it can only be the case that ‖¬¬A‖ = ‖A‖.

The upshot of these considerations may seem to be that w.r.t. classical logic, inference or incompatibility is capable of providing merely an approximation of the real thing as founded semantically. Is this a viewpoint that can be taken at face value?

Well, if we take semantics as the unquestioned background, then we are bound to do so. But this is what a proof-theoretician is not obliged to do—because, for her, truth is not a more basic concept than incompatibility and inference[15]; indeed, perhaps truth is

[15] One way to back up this perspective is to see logic as something that is constituted in terms of our discursive practices and especially of what Brandom (1994) calls the *game of giving and asking for reasons*.

best conceived as an idealized limit of provability.[16] Therefore, from this perspective, using the semantic or truth-theoretic framework as an ultimate measure of adequacy is itself ill-conceived. Thus, far from reading the results of this section as showing that "as classical logic is the logic proper, proof-theoretical foundations of logic are shaky", the proof-theoretician is likely to read them as showing that classical logic with its semantics does not fit very well with inference or proof theory and that it is not legitimate to take it as a measure of the success of inference.[17]

4 Necessity

Now consider a simple version of a necessity operator. One of the intuitions among the most straightforward version of necessity is that it should act as a closure—if A is necessarily true, then □A is true, and it is true also necessarily; while if A is not necessarily true, then □A is false, and again it is false necessarily—hence ¬□A is true necessarily. Transposed into the proof-theoretical key, □ would be such an operator if the following were the case:

$\vdash \Box A$ iff $\vdash A$; and
$\vdash \neg\Box A$ iff not $\vdash A$.

And whereas there is no problem with the first equivalence, the second one resists direct inferential treatment. The trouble is that there is no way of representing the non-inferability of A in terms of inferences.

Consider the same problem in terms of the Kripkean, possible world semantics. The first of the above equivalences would read so that □A is true in every possible world iff A is; while the second implies that □A is false in every world iff A is not true in every world. This appears to lead us to a domain of possible worlds with the universal relation of accessibility—i.e. with every world accessible from every other world. We know that the tautologies of such universal frames are axiomatized by S5; but S5 also axiomatizes the class of tautologies of a broader class of frames, namely those on which the accessibility relation is merely an equivalence (see e.g. Chellas 1980, §3.4). Hence, no axioms are capable of characterizing the universal frames, distinguishing them from all other ones; therefore, S5 is the closest we can get to our above intuitions proof-theoretically.

In fact, the logic that would conform to these intuitions would be the modal logic of Carnap (1946).[18] This is a logic, sometimes called C, which is not easy to capture in terms of inference because its theorems are not closed under substitutions (hence,

Footnote 15 continued
Proof theory would then seem to offer us the closest approach to logic's natural foundation. The point is that, when viewed like this, logic is a matter of the most general and most fundamental rules of our discursive practices. (Hence, it is not so much proof theory in the original Hilbertian sense that would be pertinent, but rather approaches to logic based upon its dialogical nature from the beginning—see, e.g., Lorenzen 1955).

16 As Dummett (1991, pp. 165) puts it: "Without doubt, the source of the concept [of truth] lies in our general conception of the linguistic practice of assertion." See also Restall (2009).

17 See Peregrin (2008).

18 See Punčochář (2012) for a discussion of this modal system and its modifications.

somebody might want to say, it is "not a real *logic*"). For example, ¬□¬A is a theorem of C whenever A is a propositional letter but not, of course, in general. The closest "well-behaved" logic is S5. (As Carnap, *ibid.*, shows, a formula is a theorem of S5 iff it is a theorem of C and each of its substitutional variants is also a theorem of C.)[19]

Now consider Brandom and Aker's definition of necessity in terms of incompatibility:

(□) ⊥X,□A iff ⊥X or there is an Y such that not ⊥X,Y and not Y ⊢ A.

First, let us note that this definition can be considerably simplified. The following two statements are proven by Brandom and Aker:

Lemma 10 *(a) ⊥□A iff there is no Y such that not ⊥Y, or there is an Y such that not ⊥Y and ⊥Y,A*

(b) ⊥X,□A iff (*⊥X or ⊥□A*)

Proof Brandom and Aker, *ibid*. 3.2 and 3.3.

Now we can prove:

Theorem 11 *In a normal structure,*

(□) ⊥X,□A iff ⊥X or not ⊢ A.*

Proof Substituting of the right-hand side of Lemma 10(a) for its left-hand side in Lemma 10(b) we get: ⊥X,□A iff ⊥X or there is an Y such that not ⊥Y and ⊥Y,A. Now it is enough to prove that not ⊢ A iff there is an Y such that not ⊥Y and ⊥Y,A. But this follows from the fact that ⊢ A iff for every Y, ⊥Y,A entails ⊥Y. □

Hence Brandom's and Aker's (*ibid.*) *prima facie* complex definition reduces to something rather simple: a compatible set is incompatible with □A iff A is not a theorem. (Hence A is a theorem iff *no* compatible set is incompatible with □A; and A is not a theorem iff *every* compatible set is incompatible with □A.) It follows that given (□), it is the case that ⊢ □A iff ⊢ A and ⊢ ¬□A iff not ⊢ A, precisely as required by our above intuition:

Theorem 12 *In any consistent normal gis complying with* (¬*K*1), (¬*K*2) *and* (□)*:*

(a) ⊢ □A iff ⊢ A; and

(b) ⊢ ¬□A iff not ⊢ A.

Proof According to Theorem 11, ⊢ A iff no set incompatible with □A is compatible, hence iff ⊢ □A. Also, not ⊢ A iff every set incompatible with □A is compatible, hence iff ⊥□A, and hence according to (BA*), iff ⊢ ¬□A. □

This indicates that by engaging incompatibility we can do full justice to our above mentioned intuition, *viz*. build the strictest, logical version of the necessity operator. If we are not able to make use of incompatibility and we must rely on inference, we

[19] Thomason (1973) offers an axiomatization of C which may be seen as capturing it in inferential terms. However, the axiomatization is based on an infinite number of axioms that apparently cannot be captured by means of a finite number of schemas.

can merely *approximate* the intuitions. Thus, here we see taking incompatibility rather than inference as a basis for making a real difference.

Let us now consider possibility:

($\Diamond$) $\perp$X,$\Diamond$A iff $\perp$X or $\perp$A.

We can show that in a normal structure, possibility defined thus can also be defined via the above necessity and negation (remember that in a normal structure, negation is classical by default):

Theorem 13 *In any consistent normal gis complying with* ($\neg K1$), ($\neg K3$), ($\Box$), *and* ($\Diamond$)*:* $\perp X,\Diamond A$ *iff* $\perp X,\neg\Box\neg A$.

Proof Assume $\perp$X,$\neg\Box\neg$A. According to ($\neg$K3$'$), this holds iff X $\vdash$ $\Box\neg$A. According to ($\vdash\perp$1), this holds iff for every Y, $\perp$Y,$\Box\neg$A entails $\perp$Y,X. Using ($\Box$*), we see that $\perp$Y,$\Box\neg$A is equivalent with $\perp$Y or not $\vdash$ $\neg$A, and hence, according to ($\neg$K3), with $\perp$Y or not $\perp$A. Thus, $\perp$X,$\neg\Box\neg$A iff for every Y, $\perp$Y or not $\perp$A entails $\perp$Y,X. As $\perp$Y certainly does entail $\perp$Y,X, this holds iff not $\perp$A entails for every Y, $\perp$Y,X, and hence iff not $\perp$A entails $\perp$X. This, in turn holds iff $\perp$A or $\perp$X, and hence iff $\perp$X,$\Diamond$A. □

This gives us the natural counterparts of (a) and (b) of Theorem 12:

Theorem 14 *In any consistent normal gis complying with* ($\neg K1$), ($\neg K2$) *and* ($\Diamond$)*:*

(a) $\vdash \Diamond A$ *iff not* $\perp A$*, and*
(b) $\vdash \neg\Diamond A$ *iff* $\perp A$.

Proof Given Theorem 12, both clauses are direct consequences of the respective clauses of Theorem 11. □

5 Conclusion

The bare bones to which logic can be reduced may be incompatibility or inference. These two concepts may be construed as interdefinable: A can be considered incompatible with B if everything is inferable from A together with B, whereas B can be considered as inferable from A iff everything incompatible with B is incompatible with A; hence in some contexts, the choice of one of the concepts as the foundation instead of the other makes no difference. However, construing the two concepts as interdefinable is not mandatory and there may be arguments against it.

In any case, the choice of the primitive term on which we choose to base our logic does bear on the nature of the ensuing logic. In particular, if we base logic on inference, then though it is intuitionistic logic that comes naturally, we can help ourselves to classical logic (though it is classical logic with "inferential" semantics); while if we choose incompatibility, we are on the way to classical logic. In some specific contexts—like the context of the very general modal logic—the choice between the two concepts may make a further difference: as incompatibility lets us articulate *non*inferability (B is *not* inferable from A if there is an X incompatible with B, but not with A), it can fare better in contexts where this is required.

From the viewpoint of semantics, both inference and incompatibility may appear incapable to more than merely approximate the truth-theoretical reality, but there is no need for the proof-theoretically minded logician to accept this perspective. Instead, she may insist that the "truth-theoretical reality" is a chimera, and that what looks like an approximation of this reality by proof-theoretical means is really an extrapolation of the proof-theoretical reality by truth-theoretical means.

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